On Performing Countably-Many Reidemeister Moves

Forest Kobayashi

April 23rd, 2021

Where are we headed?

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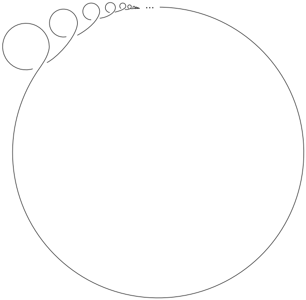
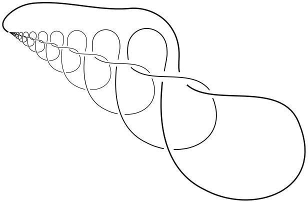


(a) What distinguishes this...



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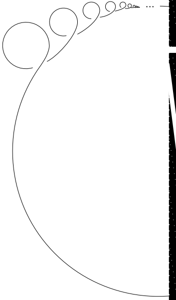

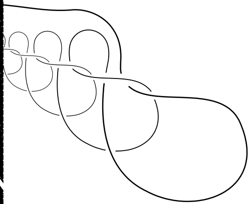

(a) What disting

(b) . . . From this?

Gameplan:

1. Intro
   - “What’s a knot?”
   - “When are knots ‘equivalent?’ How can we tell?”
2. Motivation
   - Unknotting moves & “categorification”
3. The problem
   - Tameness & wildness
   - The recipe!

What is a knot?

Definition (Informal)

Twirl a string around and “fuse” the ends.

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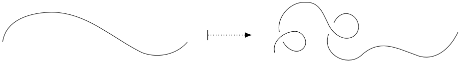

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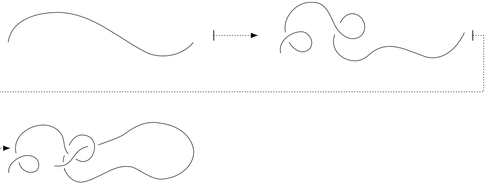

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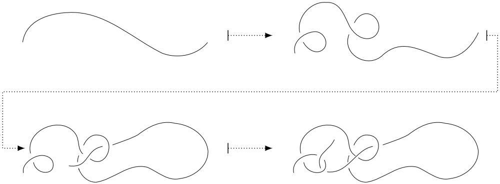

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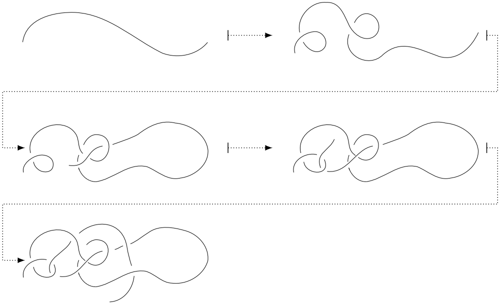

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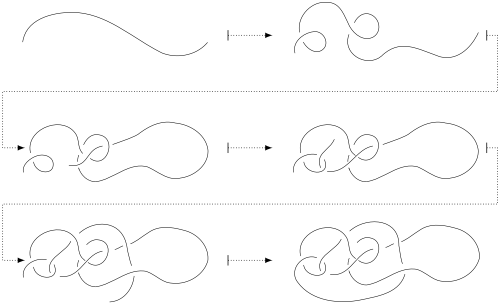

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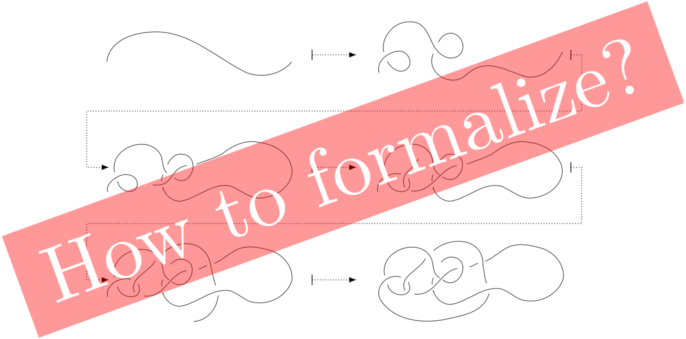

Prereq. Definition — Homeomorphism

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A *homeomorphism* is an $f : X \to Y$ such that f is bijective and continuous with f^{-1} also continuous. *(i.e. f does no cutting/gluing).*

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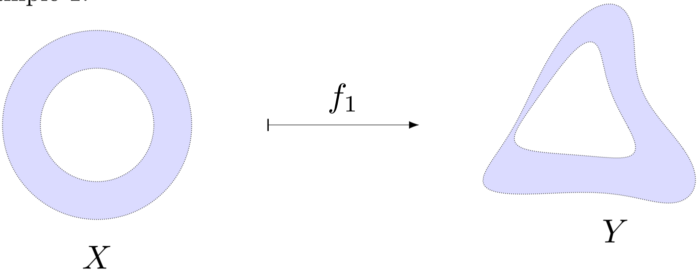

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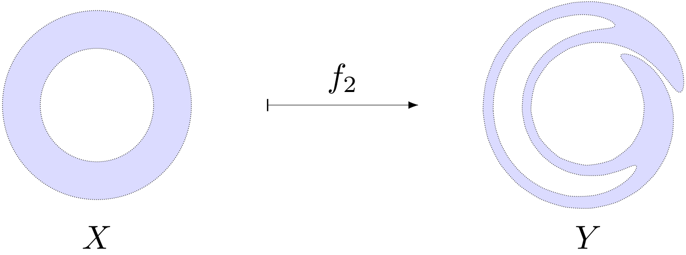

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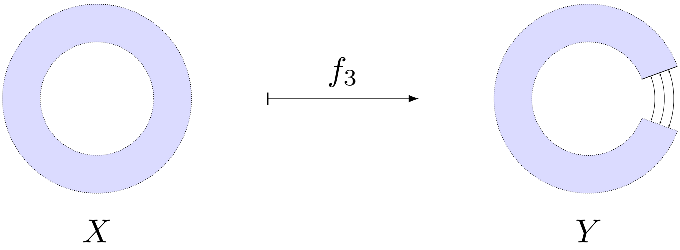

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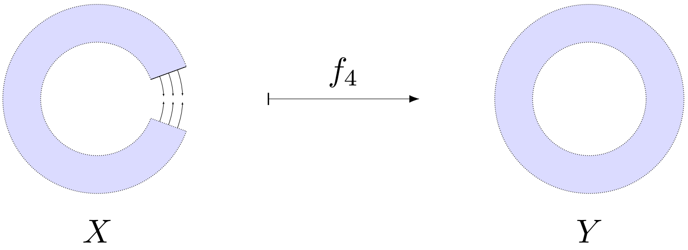
Non-example 2: “Gluing” (f^{-1} is not continuous)



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- Homeomorphisms preserve how things look "locally"
- X and Y are said to be *homeomorphic* if there's a homeomorphism $f : X \to Y$

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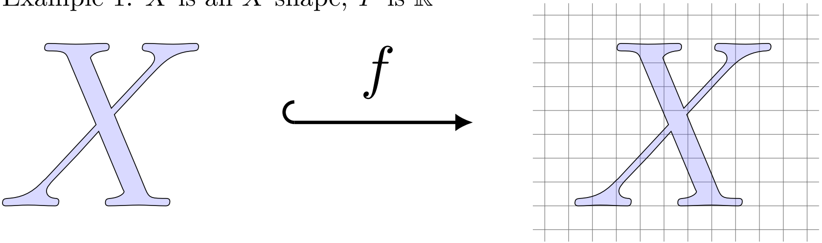
$f : X \to Y$ is an *embedding* if f is a homeomorphism between X and $f(X)$. (Since f must be injective we typically write $f : X \hookrightarrow Y$)

Example 1: X is an X shape, Y is $\mathbb{R}^2$



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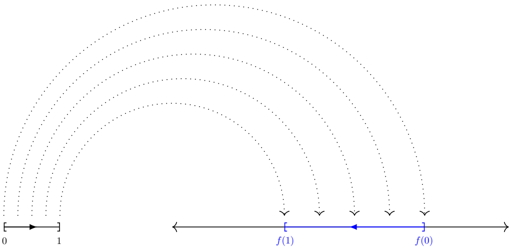

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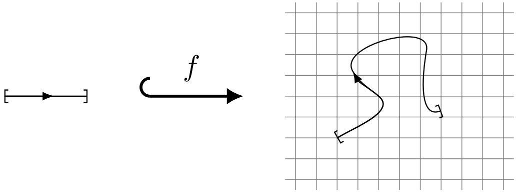

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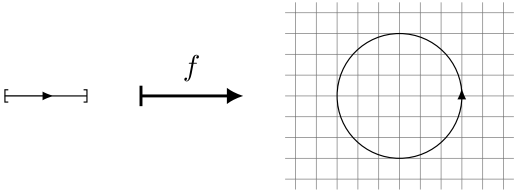
Non-example: X and $f(X)$ not homeomorphic (note the gluing!)



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- Takeaway: An embedding stuffs a copy of X into Y
- How can we use this to define knots?

Knots!

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A *knot* is an embedding $f : S^1 \hookrightarrow Y$. (For now assume $Y = \mathbb{R}^3$).

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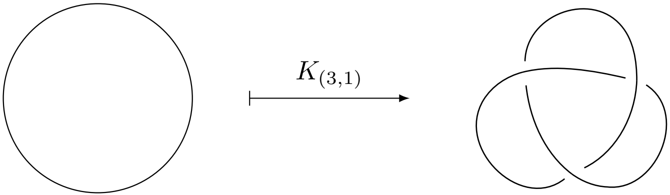

Figure: The "(3, 1)" knot

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Example 2:

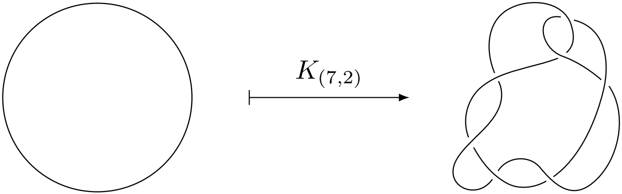

Figure: The "$(7, 2)$" knot

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Non-example 1: f is not an embedding ("cutting")

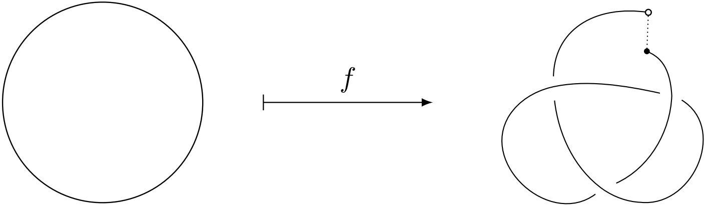

Figure: A "broken" knot

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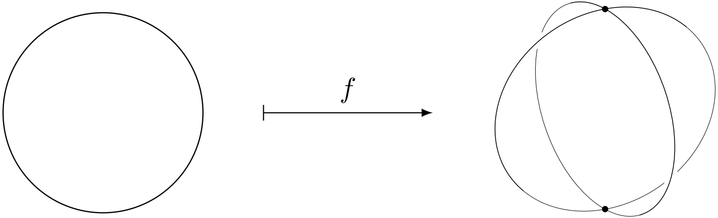

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Knot equivalence

Definition (Equivalence of Embeddings in General)

Let $f_0, f_1 : X \to Y$ be embeddings. We say that f_0 is *equivalent* to f_1 if there exists a homeomorphism $h : Y \to Y$ such that $h \circ f_0 = f_1$.

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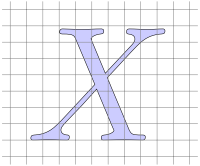
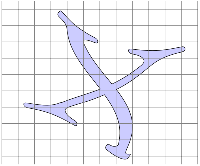
Example: Consider two embeddings of an X shape.



(a) $f_0(X)$



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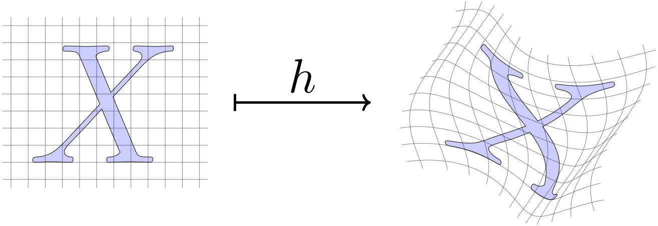
Example: These are equivalent. The h would look something like



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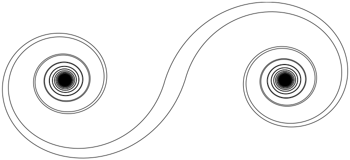
> Example 1: In $\mathbb{R}^2$, all embeddings of S^1 are equivalent. Even this can be turned into a normal circle!
>
> 

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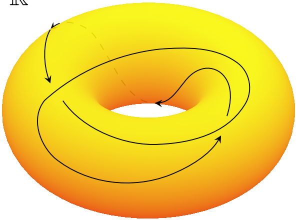
> Example 2: This embedding is "nontrivial" in a thickened torus, but not in $\mathbb{R}^3$



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> Example 3: All "nice" $f : S^1 \hookrightarrow \mathbb{R}^4$ are equivalent! (Proof: Ask at end if we have time)
>
> In fact... in most "nice" cases, knotting can only occur when $\dim(Y) - \dim(X) = 2$

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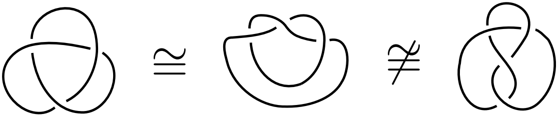
Situation for $f : S^1 \hookrightarrow \mathbb{R}^3$ is the most studied

> Example: First two are equivalent, but not to the third



Determining Equivalence: Difficulty #1

- Problem: Working with homeomorphisms explicitly is *incredibly* unergonomic.

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- Desire: A *rigorous* way to work with knots only using pictures (no equations!)
- Solution: Regular Diagrams and Reidemeister's Theorem

Regular Diagrams

Definition (Regular Diagram)

A *regular diagram* for a knot $f : S^1 \hookrightarrow \mathbb{R}^3$ has

1. Finitely-many crossing points,
2. Only two strands interacting at any given crossing,
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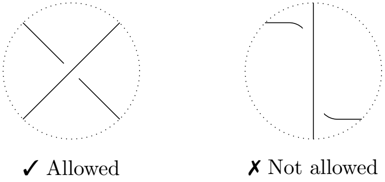

Figure: Example of axiom 1

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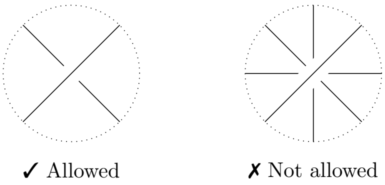

Figure: Example of axiom 2

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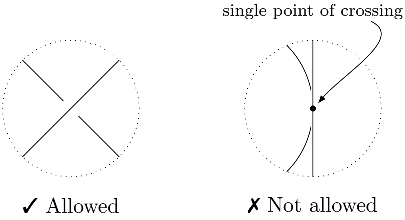

Figure: Example of axiom 3

Important note

Not every knot has a regular diagram.

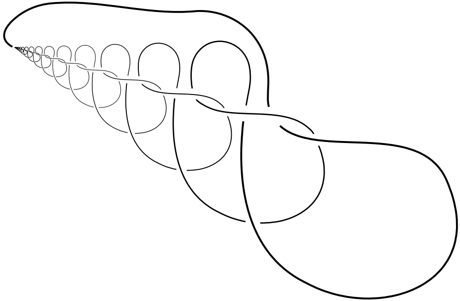

Figure: This one doesn't!

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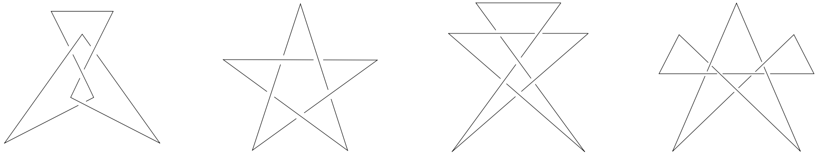

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Theorem

If $f : S^1 \hookrightarrow \mathbb{R}^3$ *is polygonal, then* f *admits a regular diagram.*

Proof: Use the finiteness

Tame & Wild Knots

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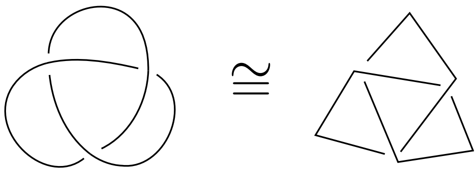
Example tame knot:



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Important property:

- Tame knots are in equivalence classes of knots with regular diagrams.
- Why does this matter? Well. . .

Almost there! Equivalence of Diagrams

Definition

We say two regular diagrams D_0, D_1 are *equivalent* iff there exist a finite sequence of the following moves taking D_0 to D_1:

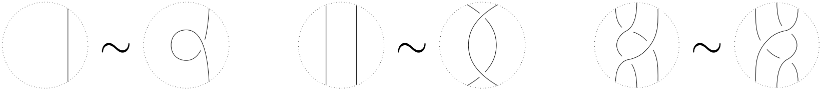

Figure: The "Reidemeister moves"

Not relevant for today, but I like to denote these by の (no; [no̞]), ゆ (yu; [jɯ̟ᵝ]), and め (me [me̞]), respectively.[1]

[1] IPA from *Wiktionary*.

Equivalence of equivalences

Theorem (Reidemeister)

Let $f_0, f_1 : S^1 \hookrightarrow \mathbb{R}^3$ be tame, and let D_0, D_1 be regular diagrams representing the equivalence classes of f_0 and f_1, respectively. Then $D_0 \cong D_1$ as diagrams iff $f_0 \cong f_1$ as embeddings.

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- Much more computationally tractable!
- . . . But actually still *incredibly* difficult for large examples (even an NP solution seems out of reach for now; [Lac16])

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- Solution?

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  2. $-\dfrac{2}{(\sqrt{47} + \frac{1}{47})^3} = 47 - \dfrac{1}{47^2}$
  3. $3x^4 + (x+3)(x^2+2x+2) + \dfrac{2}{3}(x - x^2) = 2\left(x^4 + \dfrac{3}{2}x(x^2 - 3x)\right) + 3x$

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  3. $3x^4 + (x+3)(x^2+2x+2) + \dfrac{2}{3}(x - x^2) = 2\left(x^4 + \dfrac{3}{2}x(x^2 - 3x)\right) + 3x$
  1. Left is odd, right is even
  2. Left is negative, right is positive
  3. Leading coefficients don't match

Knot Invariants

- Takeaway: Coarse heuristics can save time.
- Inspired by this:

Definition (Knot Invariant)

A *knot invariant* assigns "nice" values to knots such that equivalent knots are guaranteed to take the same value.

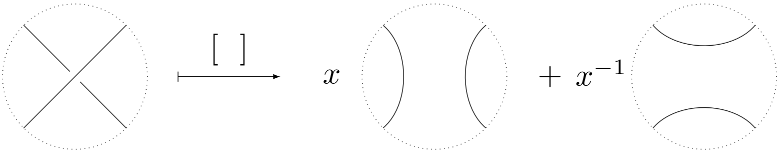
- Examples: Colouring invariants, knot polynomials, etc.

Example: Jones Polynomial

Definition (Jones Polynomial, Kauffman Bracket version)

Consider a formal polynomial in x derived from a regular diagram using the following recursive simplification process:

Rule 1:



Example: Jones Polynomial

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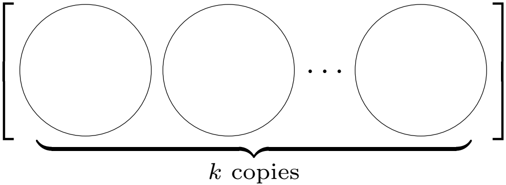

$$= \left(-x^2 - x^{-2}\right)^{k-1}$$

Example: Jones Polynomial

Definition (Jones Polynomial, Kauffman Bracket version)

Consider a formal polynomial in x derived from a regular diagram using the following recursive simplification process:

This yields a powerful invariant called the *Jones polynomial.*

What is the Jones polynomial “doing?”

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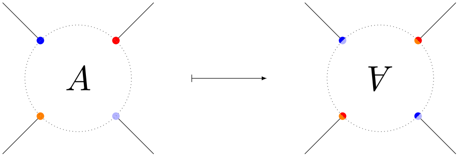
- Possibly more fruitful question: What is it *not* doing?

Definition (Mutation)

Let D_0 be a diagram. Select some region A of D_0 such that the knot intersects ∂A in four places. "Rotate" A by "180°" and call the resulting diagram D_1. This move changing D_0 into D_1 is called *mutation*.



Cont.

- The Jones polynomial *cannot distinguish between diagrams differing by a mutation.*

Cont.

- The Jones polynomial *cannot distinguish between diagrams differing by a mutation.*
- Observation: *mutations* sort of look like an action of D_4.
- Many similar rules cause problems with other invariants.
- Speculation: Can we get group structure here?

My attempt

- Use *combinatorial* encodings.

Definition

The *signed Gauss code* is a full encoding of an (oriented) n-crossing diagram using $6n$ symbols.

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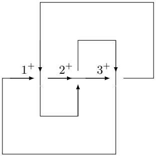
Example: $1_u^+, 2_o^+, 3_u^+, 1_o^+, 2_u^+, 3_o^+$



My attempt, cont.

- Reidemeister moves can be formulated as permutations on these strings
- …As can mutations and other similar moves.
- Typical move looks like "swap the ordering of crossing 5 and crossing 7"

The problem

- What does "swap crossing 5 and crossing 7" mean if our diagram only has 3 crossings total. . . ?
- Desire: A way to think of *all* tame knots as if they have countably-many crossings

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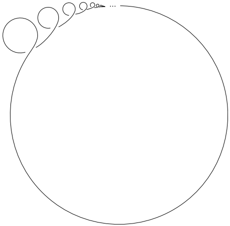
- Desire: A way to think of *all* tame knots as if they have countably-many crossings
- Solution: Add them!



How?

- Can't use Reidemeister's theorem because it assumes finiteness. Need to work directly
- Recall: Definition of equivalence

Definition (Equivalence of Embeddings in General)

Let $f_0, f_1 : X \to Y$ be embeddings. We say that f_0 is *equivalent* to f_1 if there exists a homeomorphism $h : Y \to Y$ such that $h \circ f_0 = f_1$.

- Recall: Key properties of homeomorphisms are *bijectivity* and *continuity both ways*

When life gives you metrics, make metricade

The idea is approximation. Lemmas we'll use:

Lemma

Let $(f_k)_{k=1}^{\infty}$ be a sequence of uniformly convergent continuous functions. Then $\lim_{k\to\infty} f_k$ is continuous.

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Lemma

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- Idea: Use Lemma 1 to get continuity of f in hypothesis of Lemma 2

First result (kind of silly)

Corollary

Let X be compact, and for each $k \in \mathbb{N}$, let $f_k : X \to Y$ be an embedding. Suppose that the f_k converge uniformly to some f. Then if f is injective, it's also an embedding.

Example:

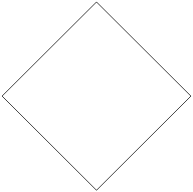

Figure: f_1

First result (kind of silly)

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Example:

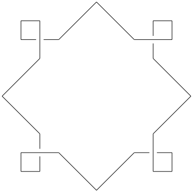

Figure: f_2

First result (kind of silly)

Corollary

Let X be compact, and for each $k \in \mathbb{N}$, let $f_k : X \to Y$ be an embedding. Suppose that the f_k converge uniformly to some f. Then if f is injective, it's also an embedding.

Example:

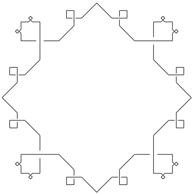

Figure: f_3

First result (kind of silly)

Corollary

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Example:

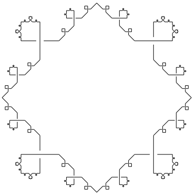

Figure: f_4

First result (kind of silly)

Corollary

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Example:

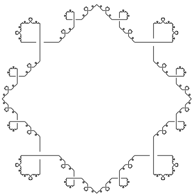

Figure: f_5

First result (kind of silly)

Corollary

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Example:

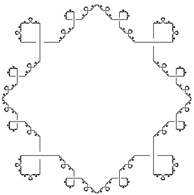


Figure: f_6

Iterative version

Theorem

Let Y be a metric space. For all $k \in \mathbb{N}$, let $h_k : Y \to Y$ be a homeomorphism and for all $n \in \mathbb{N}$, define

$$\hbar_n = \bigcirc_{k=1}^{n} h_k = (h_n \circ h_{n-1} \circ \cdots \circ h_2 \circ h_1).$$

For each k let $V_k \subseteq Y$ such that h_k is identity on V_k^c. Then provided (cont. next slide)

Iterative version

Theorem (cont.)

1. *The V_k satisfy*

$$\lim_{n\to\infty}\left(\bigcup_{k=n}^{\infty} V_k\right) = 0,$$

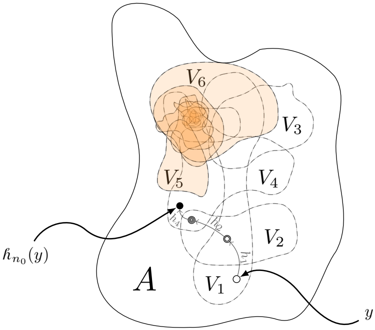
2. *There exists a compact $A \subseteq Y$ such that*

$$\left(\bigcup_{k=1}^{\infty} V_k\right) \subseteq A^\circ$$

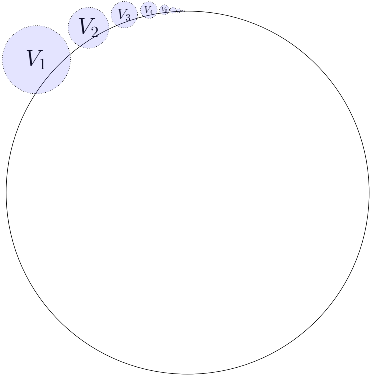
3. *h_∞ defined by $h_\infty = \lim_{n\to\infty} h_n$ exists and is bijective, then h_∞ is a homeomorphism.*

Idea of proof

- Just need to verify uniform convergence.
- The shrinking conditions on the V_k guarantee all but one point "stops moving" past some index n_0



Example 1



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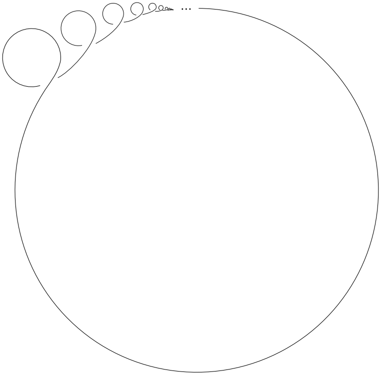
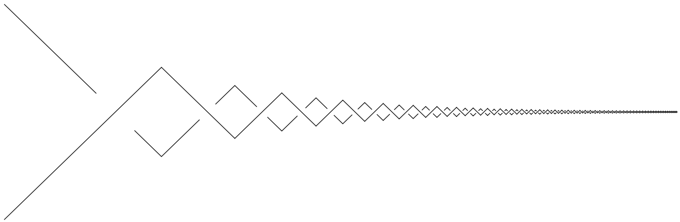
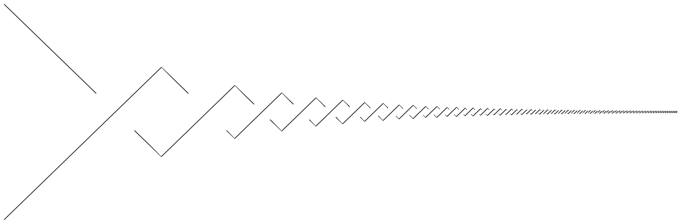


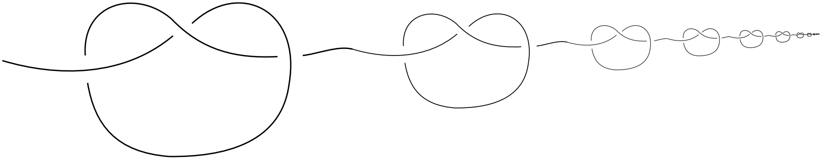
Example 2



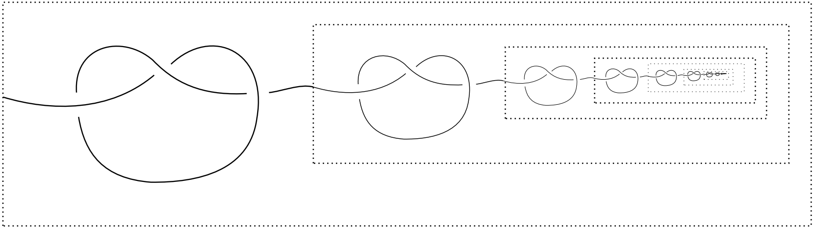
Example 3 (hard!)



Example 4



Example 4



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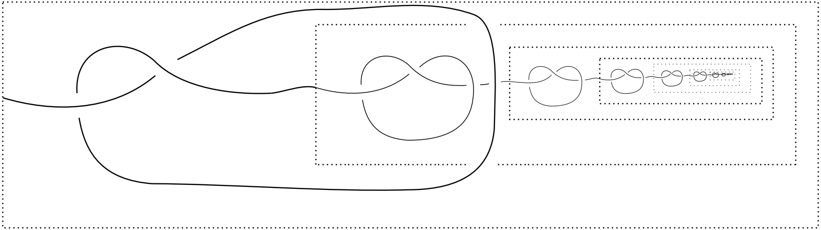

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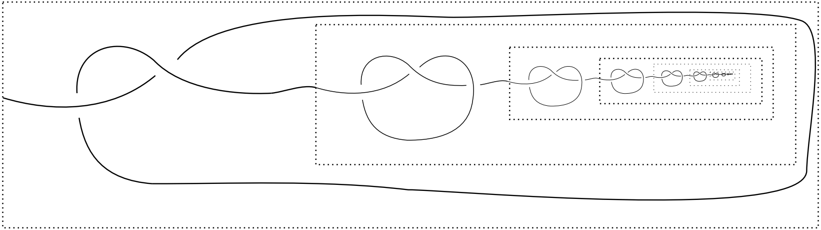

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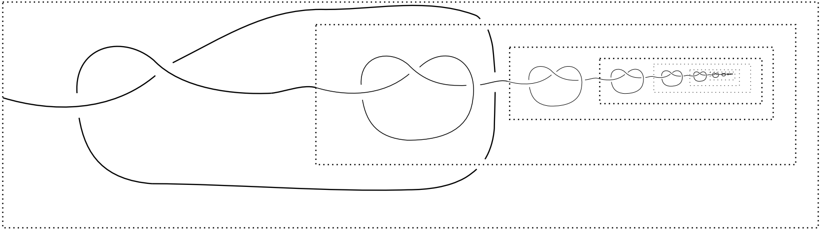

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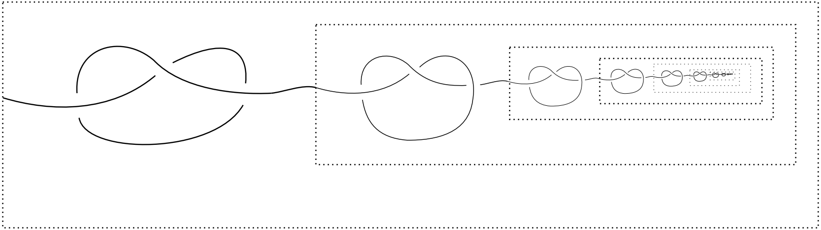

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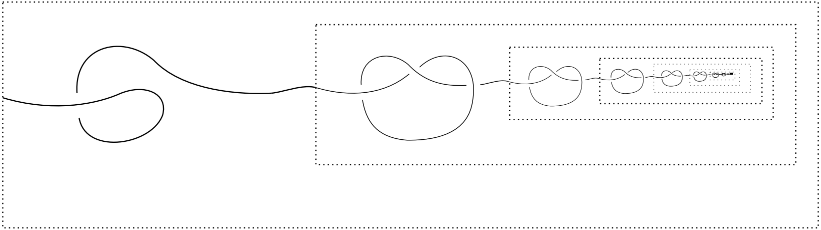

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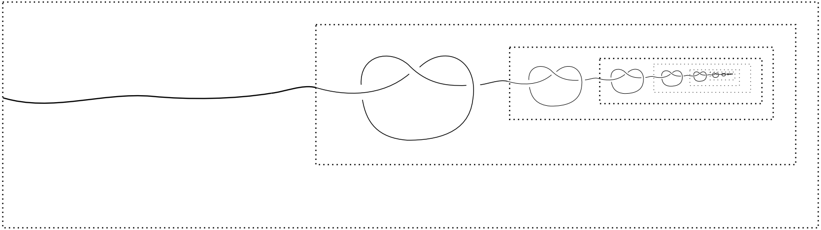
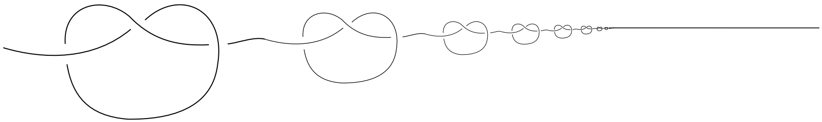
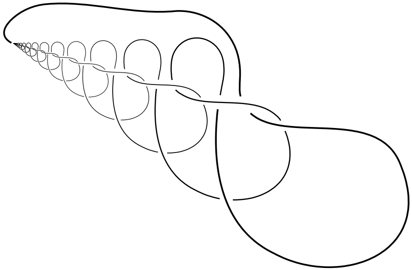


Non-example: V_k don't decay properly



Non-example: Bijectivity lost (subtle!!)



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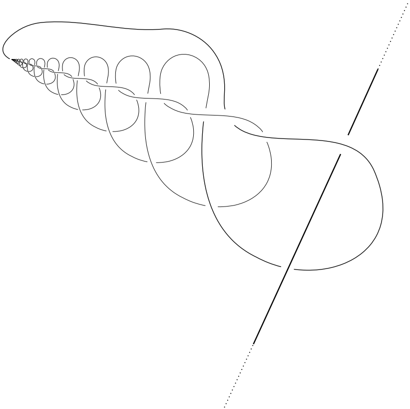

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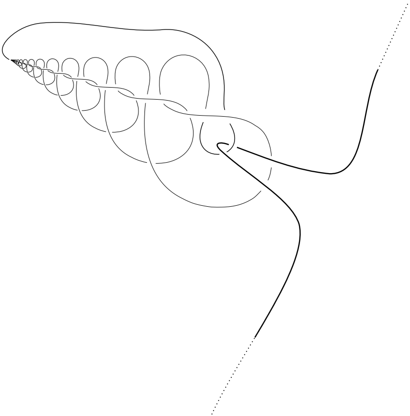

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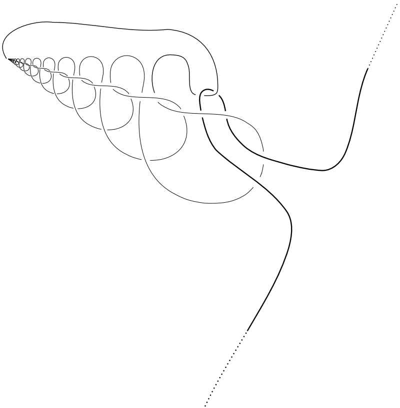

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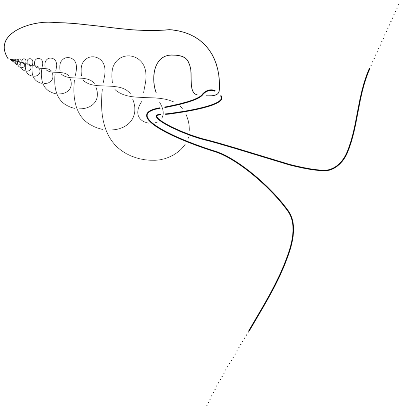

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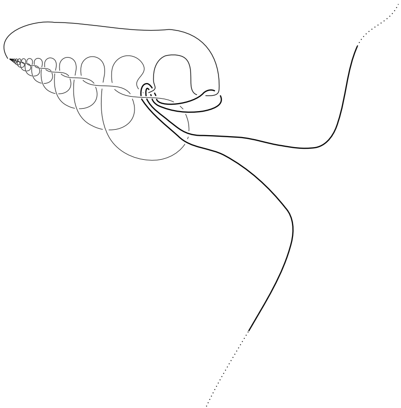

Towards a Countable Reidemeister Theorem

Definition (Discrete Diagram)

A *discrete diagram* for a knot $f : S^1 \hookrightarrow \mathbb{R}^3$ has

1. *Topologically discrete* crossing-points,
2. Only two strands intersecting at any given crossing,
3. Only "transverse" crossings.

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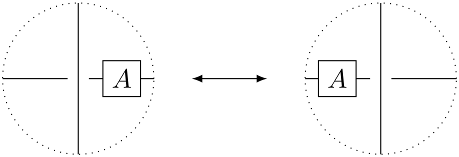
Theorem (Countable polygonal knot)

Let $f : S^1 \hookrightarrow \mathbb{R}^3$ have a discrete diagram. Then f is equivalent to a knot comprised of a countable union of straight line segments.

Proof: Unpleasant!

Conjecture!

Define the *extended Reidemeister moves* to be the original set together with a fourth move



where in the above, A is a compact set whose interior remains fixed relative to its boundary.

Let $f_0, f_1 : S^1 \hookrightarrow \mathbb{R}^3$ admit discrete diagrams D_0, D_1. Then $f_1 \cong f_2$ iff there exists a countable sequence of Reidemeister moves satisfying (slightly-modified versions of) the decay conditions on the V_k that take D_0 to D_1.

References I

- Forest D. Kobayashi, *Where the Wild Knots Are*, Bachelor's thesis, Harvey Mudd College, Claremont, CA, May 2020.
- Forest Kobayashi, *Uniform Convergence and Knot Equivalence*, arXiv (2021).
- Marc Lackenby, *Elementary knot theory*, arXiv (2016).