Constrained Wasserstein Fitting Problems

Or: "How Best to Fill a Region with a Curve?"

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The Problem

Given:

1. A region $\Omega \subseteq \mathbb{R}^2$,
2. A finite length ℓ of rope,

Find:

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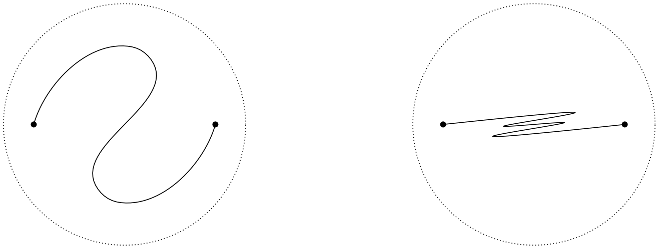

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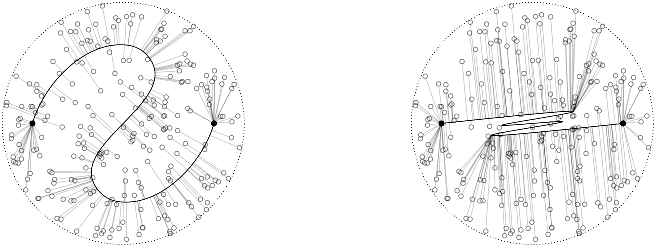

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Takeaway

Objective: Find a curve $f : [0,1] \to \Omega$ minimizing

$$\mathcal{J}_p(f) = \int_\Omega d(\omega, f([0,1]))\ d\omega$$

subject to length$(f) \leq \ell$.

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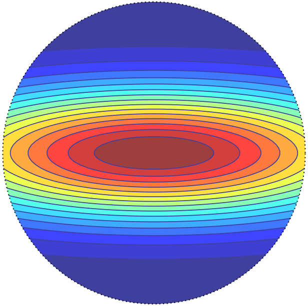
Important point:

- More proper to say we want to "fill" the Lebesgue measure $d\omega$ on Ω rather than saying we want to fill Ω.
- What if we try other measures?

Other Measures

Example: Choose $\rho \in \mathcal{P}(\Omega)$ more concentrated on the horizontal:

$$\rho \sim \mathcal{N}(0;\ [\sigma_X^2\ 0;\ 0\ \sigma_Y^2]) \qquad \sigma_Y \ll \sigma_X$$



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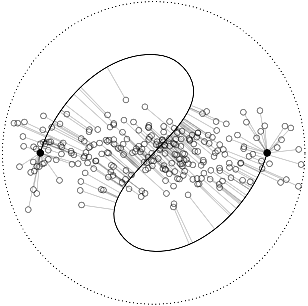

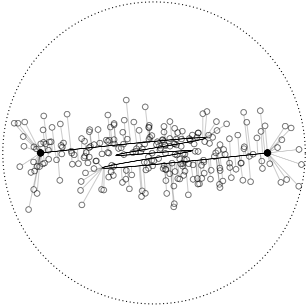

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Another important point:

- Does f need to be 1d? (No.)
- What should we make our constraint when f is e.g. a surface?

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Choice:

$$\mathcal{C}(f) = \|f\|_{W^{k,q}(X;\Omega)}$$

(assume $kq > m$ for Sobolev inequality)

Problem Statement

Given:

- $\rho \in \mathcal{P}(\Omega)$
- $\ell \geq 0$
- $p \geq 1$
- Some technical hypotheses

Minimize:

$$\mathcal{J}_p(f) = \int_\Omega \inf_{x \in X} |\omega - f(x)|^p \; d\rho(\omega)$$

Subject to:

$$\|f\|_{W^{k,q}(X;\Omega)} \leq \ell$$

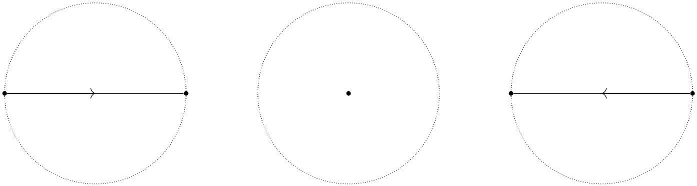
Shorthand: $J(\ell) = \inf_{\mathcal{C}(f) \leq \ell} \mathcal{J}_p(f)$

Summary of Main Results

1. Connection to OT?
2. Optimizers?
   - Exist under mild hypotheses
   - Generally nonunique (e.g. if X, Ω have symmetries)
3. Relationship between J and ℓ?
   - J continuous in ℓ
   - J (trivially) nondecreasing...more than this, hard to say.
   - Coarse asymptotic estimates from covering numbers
4. Important tool: Can find gradient of $\mathcal{J}_p$ in $C(f(X); \mathbb{R}^n)$.

Flavor of Problem: $\mathscr{J}_p$

1. $\mathscr{J}_p$ nonconvex



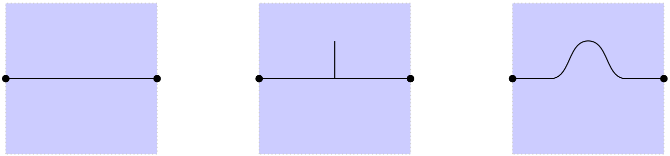
(a) f_0 (b) $f_{.5}$ (c) f_1

2. $\mathscr{J}_p$ nonconcave as well
3. But, jointly weakly continuous in f and ρ

Flavor of Problem: $\mathcal{C}(f)$

Given ε extra budget, can we improve J?

- 2^{nd}-order term makes local modifications very hard.



(a) f (b) Desired perturbation (c) Best we can do

- Can't just "extend" f past $\partial f(X)$...

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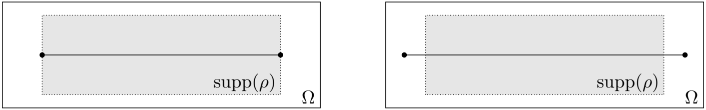

(a) $f(X)$ (b) Extended $f(X)$

Performing Local Modifications

Define a Gradient: Let $\pi_f : \Omega \to f(X)$ be closest point projection. Disintegrate ρ under π_f into $(\{\rho_y\}_{y \in f(X)}, \nu_\rho)$ and let

$$F_p(y) = p \int_{\pi_f^{-1}(\{y\})} (\omega - y)|\omega - y|^{p-2} \, d\rho_y(\omega).$$

- F_p can be discontinuous even when
  - $f \in C^\infty(X; \Omega)$
  - $f(X)$ is a C^1 manifold
- Still...$\langle F_p, \xi \rangle_{L^2}$ gives first variation in direction ξ.

Local Modification Theorems

Two versions:

- Version A: can change topology, harder-to-verify hypotheses
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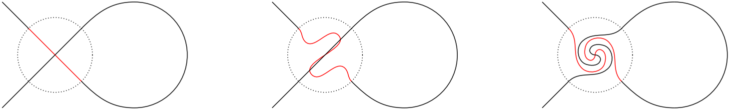
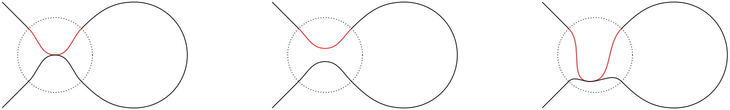


(a) Initial f (b) An f_ε for Version A (c) An f_ε for Version B



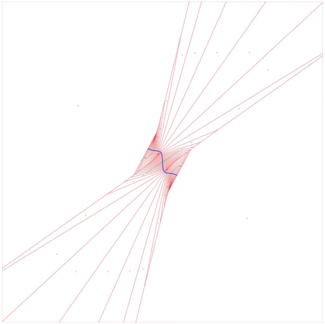
(d) Another initial f (e) An f_ε for Version A (f) An f_ε for Version B

Local Modifications, Continued

- Even though F_p potentially highly irregular...
- As long as $F_p \neq 0$ on a ν_ρ-non-null set, can find *smooth* perturbation improving $\mathcal{J}_p$!

Some Simulations

- Consistency results: Discretizing ρ, f recovers continuous solutions as resolution $\to \infty$



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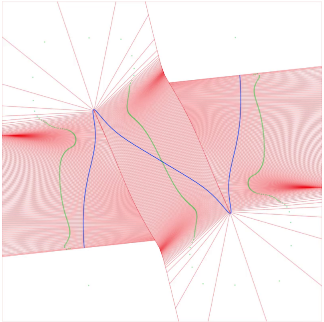

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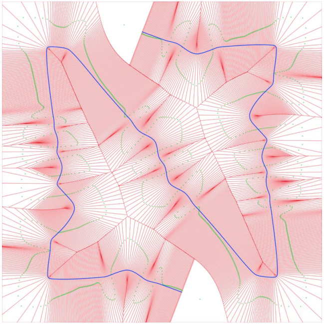

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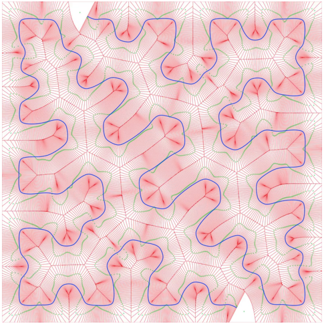

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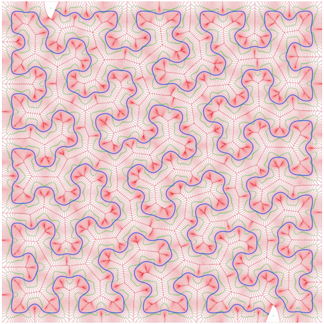

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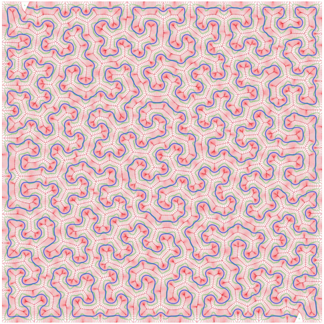

Applications

- Routing problems
- Catalyst design
- Nonlinear Dimensional Reduction
- Generative Learning

Thank you!