Monge-Kantorovich Fitting under a Sobolev Budget

Or: Sobolev-Constrained Principal Curves and Surfaces

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Joint work with Young-Heon Kim (UBC) and Jonathan Hayase (UW)

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Our Original Motivation

Given:

1. A $\rho \in \mathcal{P}(\Omega)$ ($\Omega \subseteq \mathbb{R}^n$)
2. A finite length ℓ of rope,

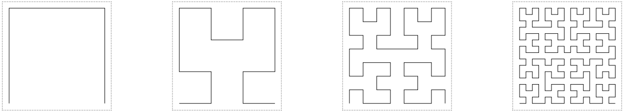
Find:

- The rope shape that best "fills" ρ.

"Filling" quantified by

$$\min_{\nu \in \mathcal{P}(f([0,1]))} \mathbb{W}_p(\rho, \nu)$$

Goal: Notion of "optimal" space-filling curves (or m-surfaces)



Optional additional regularity via Sobolev constraint

(Euclidean) Principal Curves

- Continuous $f : [0,1] \to \Omega$ with $\text{length}(f) \leq \ell$
  - (Hard constraint! C.f. soft penalty)
- Measure quality of approximation via

$$\mathcal{J}_p(f) = \int_\Omega d^p(\omega, f) \; d\rho(\omega)$$

- **Easy to show:** $\mathcal{J}_p(f) = \inf_{\nu \in \mathcal{P}(f([0,1]))} \mathbb{W}_p^p(\rho, \nu)$

(Euclidean) Principal Curves

Given:

1. A measure $\rho \in \mathcal{P}(\Omega)$,
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Find:

- Shape minimizing $\mathcal{J}_p(f) = \int_\Omega d^p(\omega, f) \; d\rho(\omega)$

Example 1: $\Omega = B_1(0; \mathbb{R}^2)$ $\rho = \mathrm{Unif}(\Omega)$ $\ell \approx .8 \cdot 2\pi$

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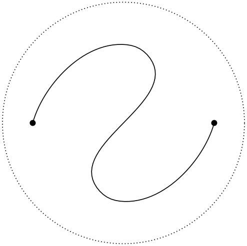
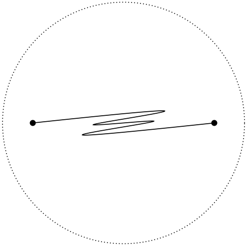


(a) A “good” filling



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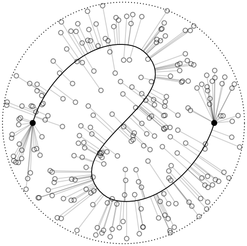

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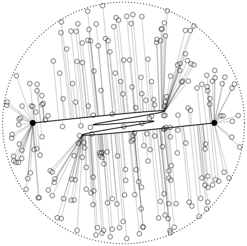

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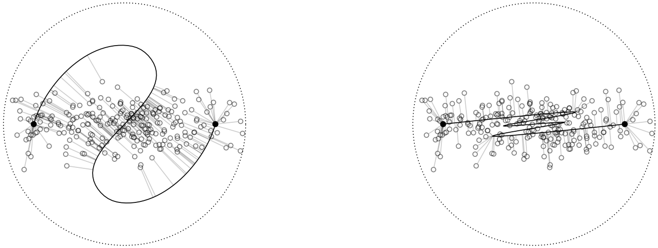

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(b) A “good” filling

Swapping length(f) for a Sobolev Norm

- Given: $f : X \to \Omega$ ($X \subseteq \mathbb{R}^m$, $\Omega \subseteq \mathbb{R}^n$)
- Choose constraint

$$\mathcal{C}(f) = \|f\|_{W^{k,q}(X;\Omega)}$$

(when $m = 1$, c.f. $\text{length}(f) = \|f'\|_{L^1(X;\Omega)}$)

- Example application: Drone routing
- Key distinction:
  - length(f) parametrization-independent.
  - For $k > 1$, $\|f\|_{W^{k,q}(X;\Omega)}$ **severely** parametrization-dependent.

Case Study: Local Improvements

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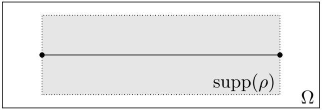
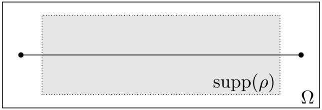


(a) $f(X)$

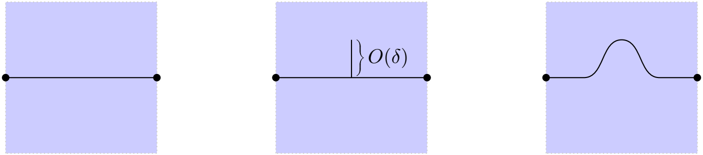


(b) Extended $f(X)$

Case Study: Local Improvements

Question: Given δ extra budget, can we improve $\mathcal{J}_p(f)$?

- Can't just "extend" f past $\partial f(X)$...
- Interior modifications? Challenging when $k > 1$



(a) Initial f (b) length(f) case (c) Sobolev case

Trying "Gradient Flow" of some kind?

Theorem

Let $f \in C(X;\Omega)$ and $\xi \in C(X;\mathbb{R}^n)$. Then if

1. $p > 1$, *or*
2. $p = 1$ *and* $\rho(f(X) \cap \operatorname{supp}(\xi)) = 0$,

then $\delta\mathcal{J}_p(f;\xi)$ can be encoded via a well-defined vector field F_ξ and measure μ_ξ on X:

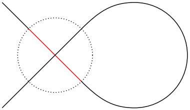
$$\delta\mathcal{J}_p(f;\xi) = \int_X -\langle F_\xi, \xi\rangle \, d\mu_\xi(x)$$

With some extra hypotheses, F_ξ, μ_ξ can be made independent of ξ.

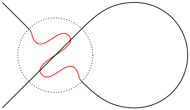
- Constraint-agnostic!
- ...But this causes problems of its own

Can handle complicated modifications

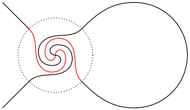
Example 1:



(a) Initial f

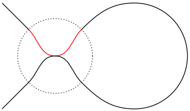


(b) Ex. perturbation 1

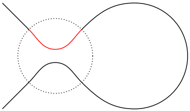


(c) Ex. perturbation 2

Example 2:



(a) Another initial f



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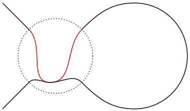
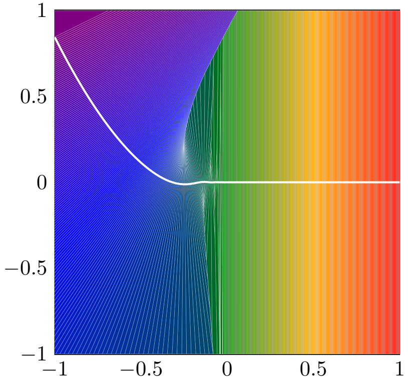


(c) Ex. perturbation 2

Problem: F_p inherits little regularity

Let $f(x) = (x, x^3 \sin(1/x))$ with Ω a square.



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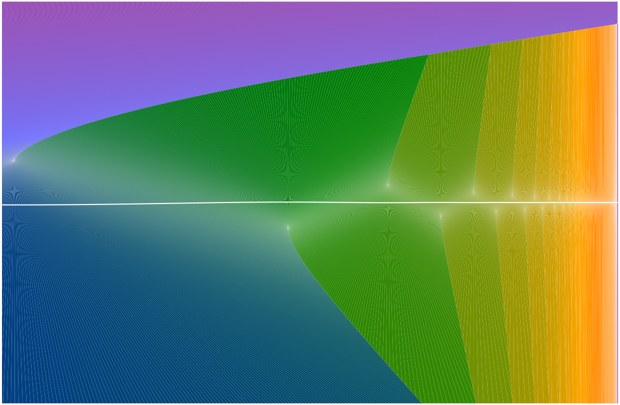
Let $f(x) = (x, x^3 \sin(1/x))$ with Ω a square(not to scale!).



Thank you!