Kaestner Brackets

UnKnot IV

Forest Kobayashi
Advisor: Sam Nelson

Harvey Mudd College

July 21st, 2019

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Agenda

- Full construction of Kaestner brackets is a bit involved, so we'll focus mostly on big-picture stuff for today:
  1. Basic definitions
  2. Why we should care about invariants
  3. Coloring invariants, skein(ish) relations, and how to combine them
  4. Distinguishing virtual knots: Kaestner Brackets
  5. Results, future directions

Recall:

Definition

A *knot* is an embedding $K : S^1 \hookrightarrow \mathbb{R}^3$.

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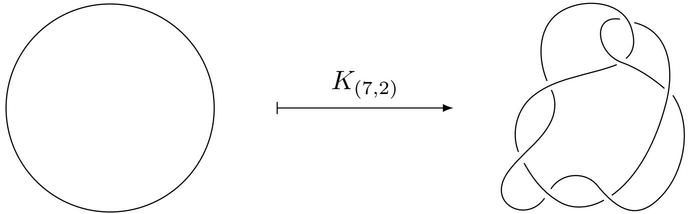

Figure 1: The $(7, 2)$ knot.

Knots can be oriented

Definition

An *oriented knot* is a knot K that has been endowed with a choice of orientation.

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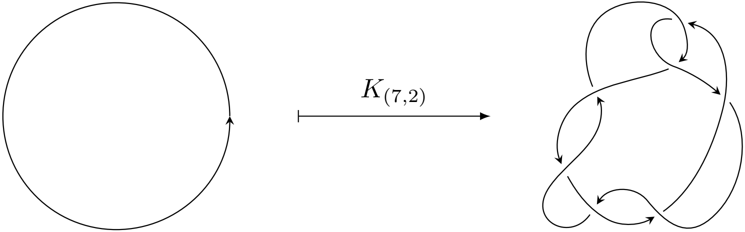
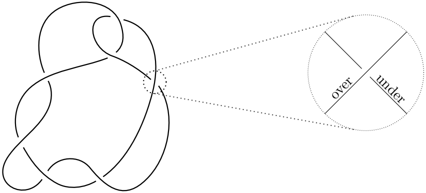


Figure 2: Oriented $(7, 2)$

Knot diagrams

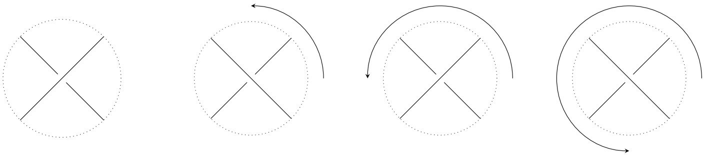
- Breaks represent crossings



- We call the top strand the "overstrand" and the bottom strand the "understrand."

A bit more on crossings

- Unoriented case: all crossings look alike!



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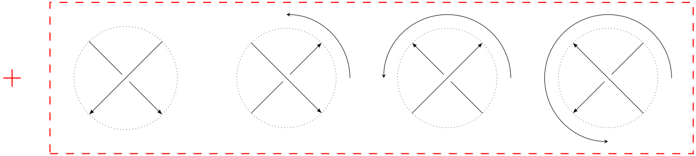

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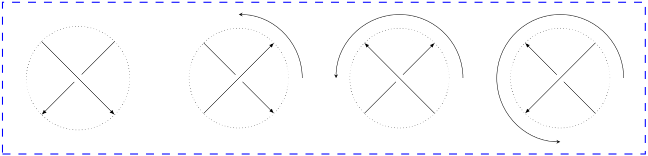
$-$ 

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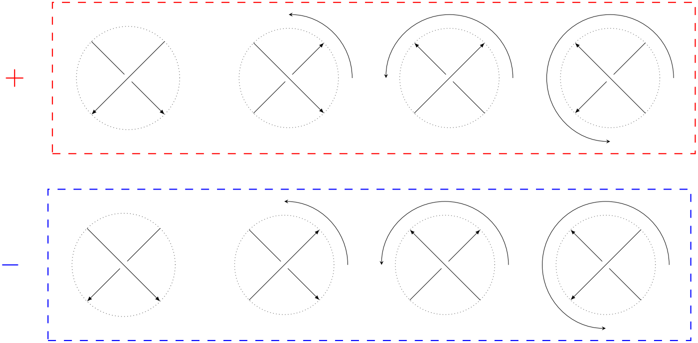

Knots reflect topological properties of ambient space

- Ex: in $\mathbb{R}^4$, all knots are unknotted!

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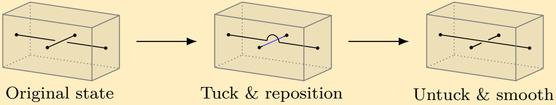
(Sketch of proof).

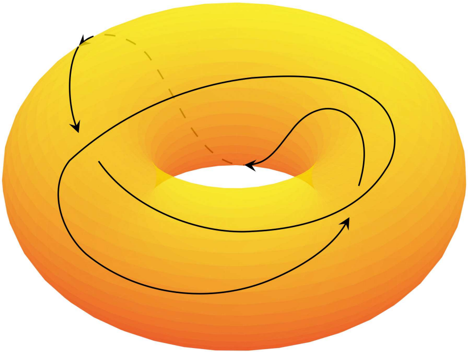
Let $K : S^1 \hookrightarrow \mathbb{R}^4$ be an embedding. At every crossing in K, we can exchange over/understrands by tucking one into the extra dimension and repositioning the other.



By a previous result, this implies K is unknotted. ∎

Knots reflect topological properties of ambient space

- Ex: virtual knots



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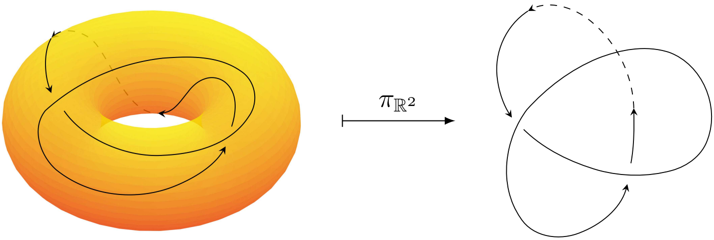

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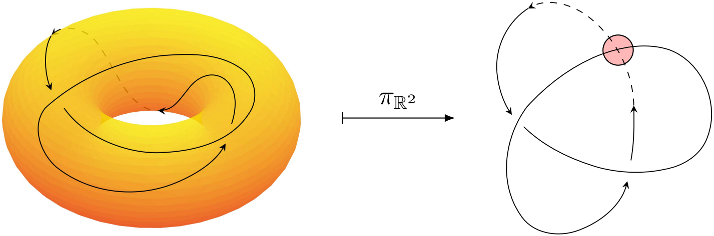
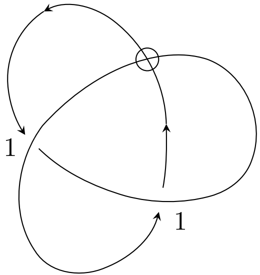


- This "crossing" in our diagram is not really there!

Knots reflect topological properties of ambient space

- Ex: virtual knots



We assign crossings in virtual knots a “parity” based on how many classical crossings we encounter travelling from understrand to overstrand

"Knot equality" needs to take this into account

Definition (Ambient Isotopy)

Let K_0, K_1 be knots. Then we say $K_0 \cong K_1$ if there is a continuous map $F : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that F_0 is identity, $K_0 \xmapsto{F_1} K_1$, and each F_t is a homeomorphism.

- Intuitively: we can deform K_0 into K_1 without tearing / gluing K_0 *or the space it lives in.*
- Various other perspectives
  - Oriented homeomorphism
  - Homotopy "through" knots

Determining equivalence

- How do we tell if $K_0 \cong K_1$?
  - Surprisingly hard!

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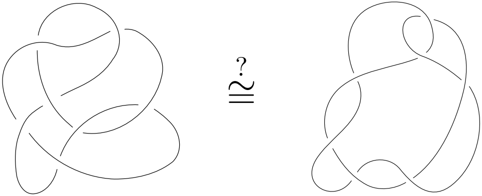

Q: Are these ambient isotopic?

Determining equivalence

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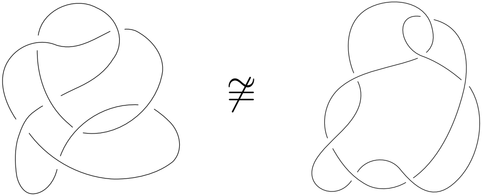

$K_{(7,6)} \ncong K_{(7,2)}$

A: Nope.

Determining equivalence

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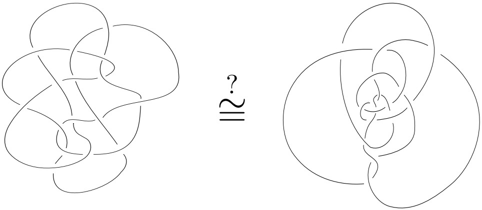
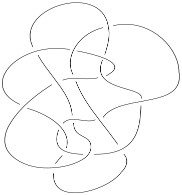
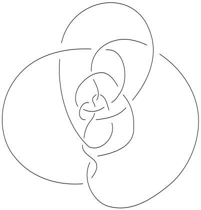


Q: What about these?

Determining equivalence

- How do we tell if $K_0 \cong K_1$?
  - Surprisingly hard!

 $\cong$ 

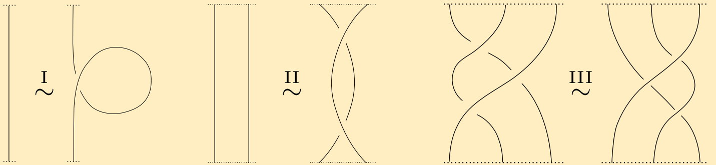
$K_{(16,32686)}$ $K_{(16,32686)}$

A: Yes!

Showing equivalence: Reidemeister moves

Theorem

$K_0 \cong K_1$ iff we can turn K_0 into K_1 via a finite sequence of the following:



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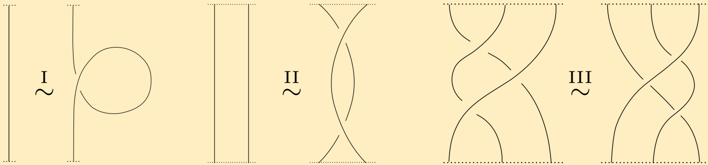

- Can prove $K_0 \cong K_1$ by exhibiting such a sequence

Showing $K_0 \not\cong K_1$ is hard

- Existing algorithms
  - Highly technical
  - Very inefficient
    - Those that have proven complexities involve things like "$O(k \uparrow\uparrow n)$ where n is large"
    - Even NP seems out-of-reach for the time being
    - See [3] for more
- What now?

Work smarter, not harder

Analogy: in 10 seconds or less. . . which of the following are true?

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1. $5\left(3^3 \cdot 11\right)^2 = 2(72 + 33 - 8)$
2. $-\dfrac{2}{\left(\sqrt{47} + \frac{1}{47}\right)^3} = 47 - \dfrac{1}{47^2}$
3. $3x^4 + (x+3)(x^2 + 2x + 2) + \dfrac{2}{3}(x - x^2) = 2\left(x^4 + \dfrac{3}{2}x(x^2 - 3x)\right) + 3x$
4. There exists no $s \in \mathbb{C}$ such that $\sum_{n=1}^{\infty} \frac{1}{n^s} = 0$, $\mathrm{Re}(s) \neq \frac{1}{2}$ and s is not a negative even integer.

Work smarter, not harder

"Clever" solutions:

1. $5\left(3^3 \cdot 11\right)^2 = 2(72+33-8)$ LHS is odd, RHS is even
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   The leading coefficients don't match
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Takeaway: sometimes we don't need to fully solve "hard" problems if we can find an easier implied problem (or, if we just quit)

Introducing: knot invariants

Definition (Knot invariant)

A *knot invariant* is a systematic ways of assigning "nice" values to knots such that equivalent knots get mapped to the same thing — i.e., a map φ such that $K_0 \cong K_1 \implies \varphi(K_0) = \varphi(K_1)$.

- Remainder mod n, sign, and leading coefficent are "invariants" in the arithmetic expressions above
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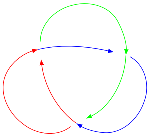
- Remainder mod n, sign, and leading coefficent are "invariants" in the arithmetic expressions above
  - Indispensable when "simplifying expressions" is hard
- Game plan for constructing knot invariants:
  1. Define algebraic structure on knots
  2. Cleverly embed in something we understand better ($\mathbb{Z}$, $\mathbb{R}[x]$, etc.)
  3. Pull back results to give us information about knots

Coloring invariants

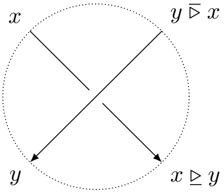
- A natural way to encode knots algebraically. Procedure:
  1. Pick a labelling set X (here, colors)
  2. Assign a label from X to each semiarc of the diagram (semiarc = portion of strand between over / under crossings)



  3. Ensure Reidemeister moves only make invertible changes to the labelling
  4. If so, this labeling scheme is an invariant of the knot

Turning it into Algebra

- Introduce two operations: $\underline{\triangleright}$, $\overline{\triangleright}$ ("under" and "over") as follows:



  - Note that we label our crossings *left to right*, not top to bottom — this makes axioms cleaner.
- Abstractly: $\underline{\triangleright}$, $\overline{\triangleright}$ encode crossing information by how it constrains the coloring

Guaranteeing invariance

- Let X be our set of labels. To guarantee invariance under the Reidemeister moves, we need the following:

  1. $\forall x \in X$, $x \,\underline{\triangleright}\, x = x \,\overline{\triangleright}\, x$
  2. $\forall x, y \in X$, the following maps are invertible: $\alpha_x(y) = y \,\overline{\triangleright}\, x$, $\beta_x(y) = y \,\underline{\triangleright}\, x$, and $S(x, y) = (y \,\overline{\triangleright}\, x, x \,\underline{\triangleright}\, y)$
  3. $\forall x, y, z \in X$, we have the following *exchange laws*:

$$(x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\underline{\triangleright}\, y) = (x \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z)$$
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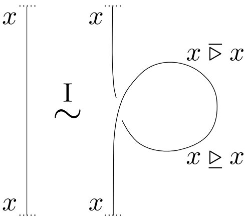
- Such a labelling scheme is called a *biquandle*.

Connection to Reidemeister moves

- Axiom 1 is required by Reidemeister I, Axiom 2 by Reidemeister II, and Axiom 3 by Reidemeister III
- Reidemeister I:

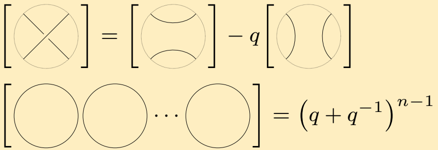


- Similarly for Reidemeister 2, Reidemeister 3

Recall: skein(ish) relations

Ex: Jones Polynomial.

Let [ ] satisfy



Then define the *Jones Polynomial* by

$$J(L) = (-1)^n q^{p-2n} \langle L \rangle$$

where n is the number of negative crossings, and p is the number of positive crossings. ([1], [4]) ■

Pros & Cons

- Pros:
  - Gives us polynomials, which are often easier to work with than birack-flavored invariants
  - Can use Reidemeister moves on intermediate smoothing states
- Cons:
  - Geometric interpretation can be challenging
  - Requires recursive enumeration of smoothed states, which is $O(2^n)$

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  - Requires recursive enumeration of smoothed states, which is $O(2^n)$
  - (k)not quandle-y enough

Coloring-dependent skein(ish) coefficients

Definition (Biquandle Brackets)

Let X be a biquandle, and R a commutative ring with identity. Let $w \in R^\times$, $\delta \in R$, and $A, B : X \times X \to R^\times$ such that

1. $\forall x \in X$,
$$A_{x,x}^2 B_{x,x}^{-1} = w$$
2. $\forall x, y \in X$,
$$-A_{x,y}^{-1} B_{x,y} - A_{x,y} B_{x,y}^{-1} = \delta$$
3. (cont. on next slide)

N.B. — for the sake of space, we write $A_{x,y}$ (or sometimes $A_{\substack{x\\y}}$) in place of $A(x, y)$.

Coloring-dependent skein(ish) coefficients

Definition

3. $\forall x, y, z \in X$,

$$A_{\substack{x\\y}} \cdot A_{\substack{x \,\underline{\triangleright}\, y\\ z \,\overline{\triangleright}\, y}} \cdot A_{\substack{y\\z}} = A_{\substack{y \,\overline{\triangleright}\, x\\ z \,\overline{\triangleright}\, x}} \cdot A_{\substack{x\\z}} \cdot A_{\substack{x \,\underline{\triangleright}\, z\\ y \,\underline{\triangleright}\, z}}$$

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(cont. on next slide)

Coloring-dependent skein(ish) coefficients

Definition

3. (cont.)

$$A_{\substack{x\\y}} \cdot B_{\substack{x \,\underline{\triangleright}\, y\\ z \,\overline{\triangleright}\, y}} \cdot A_{\substack{y\\z}} = A_{\substack{y \,\overline{\triangleright}\, x\\ z \,\overline{\triangleright}\, x}} \cdot A_{\substack{x\\z}} \cdot B_{\substack{x \,\underline{\triangleright}\, z\\ y \,\underline{\triangleright}\, z}} + B_{\substack{y \,\overline{\triangleright}\, x\\ z \,\overline{\triangleright}\, x}} \cdot A_{\substack{x\\z}} \cdot A_{\substack{x \,\underline{\triangleright}\, z\\ y \,\underline{\triangleright}\, z}}$$
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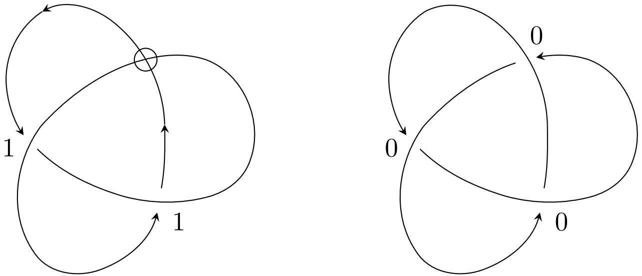
$$+ \delta B_{\substack{x\\y}} \cdot A_{\substack{x \,\underline{\triangleright}\, y\\ z \,\overline{\triangleright}\, y}} \cdot B_{\substack{y\\z}} + B_{\substack{x\\y}} \cdot B_{\substack{x \,\underline{\triangleright}\, y\\ z \,\overline{\triangleright}\, y}} \cdot B_{\substack{y\\z}}$$

Cool facts about biquandle brackets

- Interpretation of axioms:
  - w^{n-p} = writhe correction factor
  - δ adjusts for when we introduce new components
  - Axiom 3 reflects the smoothings of Reidemeister III
- Well-known special cases
  - Many biquandle invariants
  - Jones, HOMFLYPT polynomials
- Intuitive summary: biquandle brackets move structure off of strand labels and into coefficients

Encorporating parity

- Recall: in virtual knots, we can assign parity to crossings



- How do we distinguish these?

- Idea: make $\overline{\triangleright}$, $\underline{\triangleright}$ functions depend on parity of crossing
- How do we adapt our biquandle definition?
  - Crossings in Reidemeister I moves are always even, so no constraints there
  - Reidemeister II still forces invertibility
  - Reidemeister III?

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Lemma (Nelson et. al, [2])

Let K be a virtual knot. Then for any Reidemeister III move, either

1. *All of the crossings are even, or*
2. *Two are odd and one is even*

  - Hence. . .

Parity Biquandles

Definition (Nelson et. al)

Let X be a set of labels together with four binary operations: $\overline{\triangleright}^0, \underline{\triangleright}^0, \overline{\triangleright}^1, \underline{\triangleright}^1$ such that

1. X together with $\overline{\triangleright}^0, \underline{\triangleright}^0$ is a biquandle (X with $\overline{\triangleright}^1, \underline{\triangleright}^1$ need not be)
2. Biquandle axiom 2 applies to $\overline{\triangleright}^1, \underline{\triangleright}^1$
3. For all $(a, b, c) \in \{(1,1,0), (1,0,1), (0,1,1)\}$ and for all $x, y, z \in X$, we have

$$(z \overline{\triangleright}^a y) \overline{\triangleright}^b (x \overline{\triangleright}^c y) = (z \overline{\triangleright}^b x) \overline{\triangleright}^a (y \underline{\triangleright}^c x)$$

$$(x \overline{\triangleright}^a y) \underline{\triangleright}^b (z \overline{\triangleright}^c y) = (x \underline{\triangleright}^b z) \overline{\triangleright}^a (y \underline{\triangleright}^c z)$$

$$(y \underline{\triangleright}^a x) \underline{\triangleright}^b (z \overline{\triangleright}^c x) = (y \underline{\triangleright}^b z) \underline{\triangleright}^a (x \underline{\triangleright}^c z)$$

and finally...Kaestner Brackets!

Definition

Let $(X, \overline{\triangleright}^0, \underline{\triangleright}^0, \overline{\triangleright}^1, \underline{\triangleright}^1)$ be a parity biquandle, and let R be a commutative ring with identity. Let $\delta \in R$ and $A_0, B_0, A_1, B_1 : X \times X \to R^\times$. Then we call $((X), A_0, B_0, A_1, B_1)$ a *Kaestner bracket* iff the following hold:

1. $((X, \overline{\triangleright}^0, \underline{\triangleright}^0), A_0, B_0)$ is a biquandle bracket,
2. A_1, B_1 are invertible,
3. For all $x, y \in X$,

$$\delta = -A_{1,x,y} \cdot B_{1,x,y}^{-1} - A_{1,x,y}^{-1} \cdot B_{1,x,y}$$

(cont. on next slide)

and finally. . . Kaestner Brackets!

Definition

3. (cont.) For all $(a, b, c) \in \{(1,1,0), (1,0,1), (0,1,1)\}$, we have the following:

$$\overbrace{A_{a,\substack{x\\y}} \cdot A_{b,\substack{x \underline{\triangleright}^{a} y\\ z \overline{\triangleright}^{c} y}} \cdot A_{c,\substack{y\\z}}}^{\text{(i)}} = \overbrace{A_{c,\substack{y \overline{\triangleright}^{a} x\\ z \overline{\triangleright}^{b} x}} \cdot A_{b,\substack{x\\z}} \cdot A_{a,\substack{x \underline{\triangleright}^{b} z\\ y \underline{\triangleright}^{c} z}}}^{\text{(I)}}$$

$$\overbrace{A_{a,\substack{x\\y}} \cdot B_{b,\substack{x \underline{\triangleright}^{a} y\\ z \overline{\triangleright}^{c} y}} \cdot B_{c,\substack{y\\z}}}^{\text{(iv)}} = \overbrace{B_{c,\substack{y \overline{\triangleright}^{a} x\\ z \overline{\triangleright}^{b} x}} \cdot B_{b,\substack{x\\z}} \cdot A_{a,\substack{x \underline{\triangleright}^{b} z\\ y \underline{\triangleright}^{c} z}}}^{\text{(VII)}}$$

$$\overbrace{B_{a,\substack{x\\y}} \cdot B_{b,\substack{x \underline{\triangleright}^{a} y\\ z \overline{\triangleright}^{c} y}} \cdot A_{c,\substack{y\\z}}}^{\text{(vii)}} = \overbrace{A_{c,\substack{y \overline{\triangleright}^{a} x\\ z \overline{\triangleright}^{b} x}} \cdot B_{b,\substack{x\\z}} \cdot B_{a,\substack{x \underline{\triangleright}^{b} z\\ y \underline{\triangleright}^{c} z}}}^{\text{(IV)}}$$

and finally…Kaestner Brackets!

Definition

$$\overbrace{A_{a,\substack{x\\ y}} \cdot B_{b,\substack{x\,\underline{\triangleright}^{a}\,y\\ z\,\overline{\triangleright}^{c}\,y}} \cdot A_{c,\substack{y\\ z}}}^{\text{(iii)}} = \overbrace{A_{c,\substack{y\,\overline{\triangleright}^{a}\,x\\ z\,\overline{\triangleright}^{b}\,x}} \cdot A_{b,\substack{x\\ z}} \cdot B_{a,\substack{x\,\underline{\triangleright}^{b}\,z\\ y\,\underline{\triangleright}^{c}\,z}}}^{\text{(II)}} + \overbrace{B_{c,\substack{y\,\overline{\triangleright}^{a}\,x\\ z\,\overline{\triangleright}^{b}\,x}} \cdot A_{b,\substack{x\\ z}} \cdot A_{a,\substack{x\,\underline{\triangleright}^{b}\,z\\ y\,\underline{\triangleright}^{c}\,z}}}^{\text{(V)}}$$

$$+ \underbrace{\delta B_{c,\substack{y\,\overline{\triangleright}^{a}\,x\\ z\,\overline{\triangleright}^{b}\,x}} \cdot A_{b,\substack{x\\ z}} \cdot B_{a,\substack{x\,\underline{\triangleright}^{b}\,z\\ y\,\underline{\triangleright}^{c}\,z}}}_{\text{(VI)}} + \underbrace{B_{c,\substack{y\,\overline{\triangleright}^{a}\,x\\ z\,\overline{\triangleright}^{b}\,x}} \cdot B_{b,\substack{x\\ z}} \cdot B_{a,\substack{x\,\underline{\triangleright}^{b}\,z\\ y\,\underline{\triangleright}^{c}\,z}}}_{\text{(VIII)}}$$

$$\overbrace{A_{c,\substack{y\,\overline{\triangleright}^{a}\,x\\ z\,\overline{\triangleright}^{b}\,x}} \cdot B_{b,\substack{x\\ z}} \cdot A_{a,\substack{x\,\underline{\triangleright}^{b}\,z\\ y\,\underline{\triangleright}^{c}\,z}}}^{\text{(III)}} = \overbrace{A_{a,\substack{x\\ y}} \cdot A_{b,\substack{x\,\underline{\triangleright}^{a}\,y\\ z\,\overline{\triangleright}^{c}\,y}} \cdot B_{c,\substack{y\\ z}}}^{\text{(ii)}} + \overbrace{B_{a,\substack{x\\ y}} \cdot A_{b,\substack{x\,\underline{\triangleright}^{a}\,y\\ z\,\overline{\triangleright}^{c}\,y}} \cdot A_{c,\substack{y\\ z}}}^{\text{(v)}}$$

$$+ \underbrace{\delta B_{a,\substack{x\\ y}} \cdot A_{b,\substack{x\,\underline{\triangleright}^{a}\,y\\ z\,\overline{\triangleright}^{c}\,y}} \cdot B_{c,\substack{y\\ z}}}_{\text{(vi)}} + \underbrace{B_{a,\substack{x\\ y}} \cdot B_{b,\substack{x\,\underline{\triangleright}^{a}\,y\\ z\,\overline{\triangleright}^{c}\,y}} \cdot B_{c,\substack{y\\ z}}}_{\text{(vii)}}$$

Corresponding smoothings

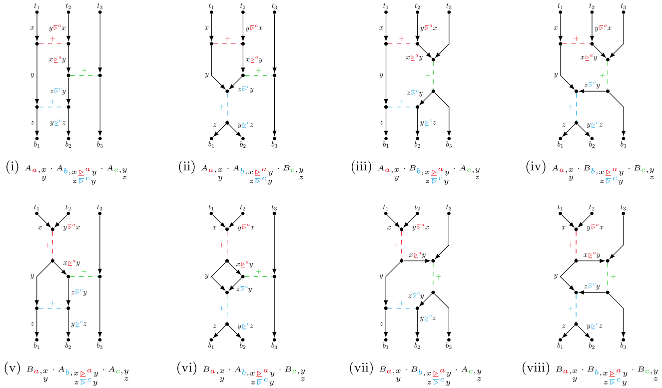


Figure 4: LHS Smoothings

Corresponding smoothings

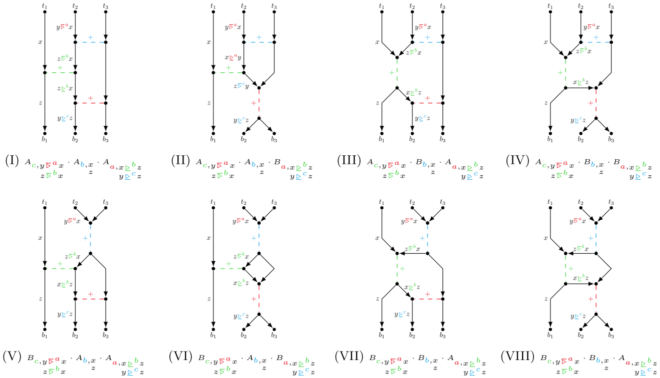


Figure 4: RHS Smoothings

Results

- The choice we had:
  - (a) Pursue further results theoretically
  - (b) Pursue further results computationally

Results

- The choice we had:
  (a) Pursue further results theoretically
  (b) Pursue further results computationally
- Decision: both, but start with (b) first

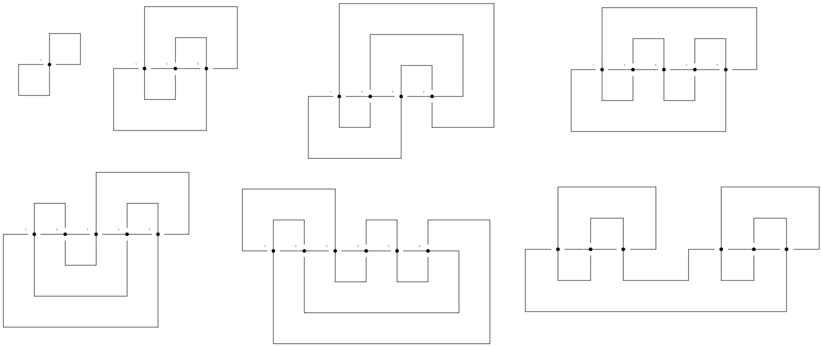
Computational results

- Huge algorithmic improvements to search code for biquandles, parity biquandles, and biquandle brackets
- Performance comparison (new code vs. old code):
  - On first non-instant return: ≈ 1 sec vs. ≈ 301 sec
  - On a previously unfeasible computation: ≈ 46 sec vs. >50 day runtime (this is lower bound is very conservative)
- Plus, first examples of Kaestner brackets!
- Possible reduction to graph alg

Future work

- Implement the graph-based search algorithm
- Categorification of knots (inspired by δ)
  - Current idea: we have lots of known unknotting moves — why not use them as knotting moves instead?
  - The UnKnot becomes the “identity” (!)
- A new knot presentation!

A new grid presentation!



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References

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