Kaestner Brackets

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Introduction

Definition 1. Let X be a topological space. A *knot* in X is an embedding $K : S^1 \hookrightarrow X$.

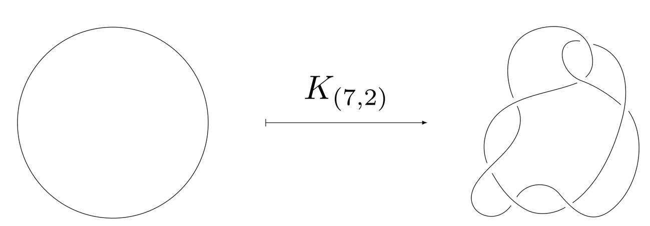


Figure 1: Example knot

Usually, we assume $X = \mathbb{R}^3$ (*classical knots*), but we may also consider knots in thickened orientable surfaces (*virtual knots*). In any case, we want to talk about what it means for two knots to be "the same." This turns out to depend on the choice of X.

- All knots in $\mathbb{R}^3$ come unknotted in $\mathbb{R}^4$.
- Some knots on a torus come unknotted in $\mathbb{R}^3$.
- Definition of knot equivalence should reflect this.

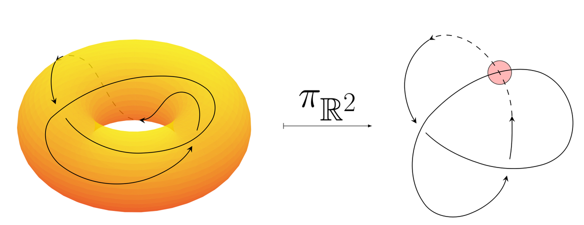
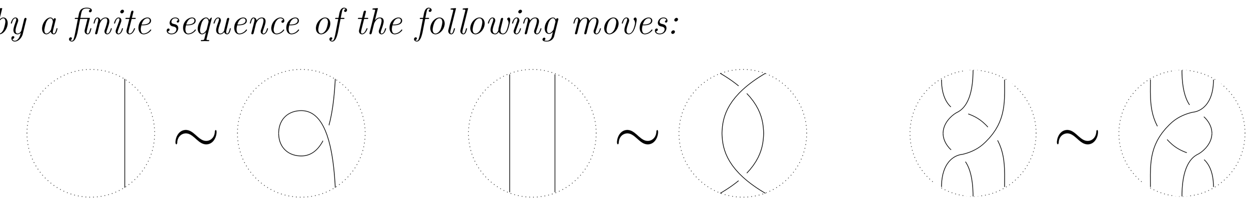


Figure 2: Knot on torus that comes undone in $\mathbb{R}^3$

Definition 2 (Ambient Isotopy)**.** Let K_0, K_1 be knots in X. Then we say $K_0 \cong K_1$ if there is a continuous map $F : X \times [0,1] \to X$ such that $F(K_0, 0) = K_0$, $F(K_0, 1) = K_1$, and each $F(\cdot, t)$ is a homeomorphism.

- If we restrict ourselves to certain well-behaved knots, we can work entirely through diagrams.

Theorem 1 (Reidemeister, 1927)**.** *Two (tame) knots K_0, K_1 are equivalent iff their diagrams are related by a finite sequence of the following moves:*



- Very elegant characterization of knot equivalence on a theoretical level.
- However: in practice, Reidemeister-based algorithms are very inefficient (not even NP).

Knot Invariants

Motivation: in 20 seconds or less, which are true?	**Clever Solutions**:
1. $5(3^3 \cdot 11)^2 = 2(72 + 33 - 8)$	1. LHS odd, RHS even
2. $-\dfrac{2}{(\sqrt{47} + \frac{1}{47})^3} = 47 - \dfrac{1}{47^2}$	2. LHS negative, RHS positive
3. $3x^4 + (x+3)(x^2 + 2x + 2) + \frac{2}{3}(x - x^2) = 2\left(x^4 + \frac{3}{2}x(x^2 - 3x)\right) + 3x$	3. Leading coefficients don't match

Idea: Coarse heuristics sometimes let us eschew full computations. Let's apply this to knots.

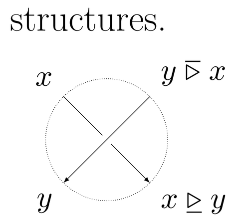
Definition 3 (Knot Invariant)**.** Let **Knot** be the category of knots, and let **C** be another category. Then a *knot invariant* is a map $F : \mathbf{Knot} \to \mathbf{C}$ such that

$$K_0 \cong K_1 \implies F(K_0) \cong F(K_1).$$

- Intuitively: a systematic way of assigning "nice" values to knots such that equivalent knots get mapped to the same thing.
- "Clever solutions" above can be thought of as analogous invariants for strings of arithmetic expressions in $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{Q}[x]$, respectively.
- Today: two important classes of knot invariants, *coloring invariants* and *skein-based invariants*.

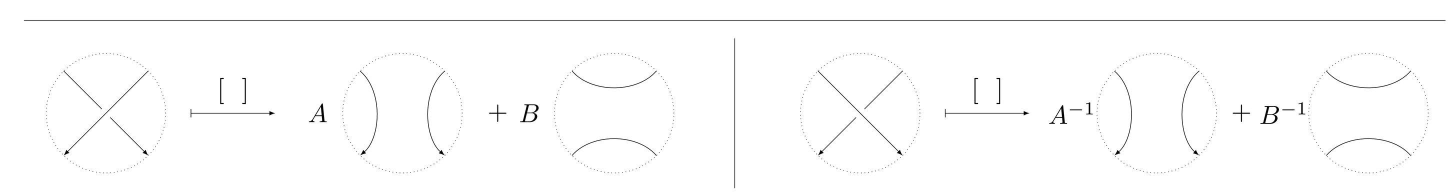
Coloring invariants: Loosely speaking, these encode knots in group-like algebraic structures.

Example (Biquandles): Let X be a set and K be a knot represented by some diagram D. Label ("color") each arc in D by an element of X. Then, define two binary operations $\underline{\triangleright}, \overline{\triangleright}$ (read "under" and "over," respectively) that describe how our labels change when strands cross (see diagram at right).



By translating the Reidemeister moves into algebraic axioms for $\underline{\triangleright}, \overline{\triangleright}$, we can turn "coloring by X" into a knot invariant! In this case we call $(X, \underline{\triangleright}, \overline{\triangleright})$ a *biquandle*.

Skein-based invariants: These recursively convert diagrams to polynomials by a "bracket map" like the following (the process terminates with unknots being assigned a fixed value δ and then multiplying by a normalization constant to account for *writhe*).



Example (Jones Polynomial): The celebrated *Jones Polynomial* can be constructed this way using the *Kauffman Bracket*, which corresponds to the choices $B = A^{-1}$, and $\delta = -(A^2 + A^{-2})$.

Some notes:

- Many of our best invariants are skein-based. How can we make them even stronger?
- One approach: the skein rules treat all crossings in a knot as if they are "the same." To distinguish them, we can first color the knot with a biquandle and then make the coefficients A, B dependent on the coloring at each crossing. This yields *biquandle brackets*, which were first introduced in [3].

Kaestner Brackets

Building off this work, we introduce a generalization of *biquandle brackets* called *Kaestner brackets*. These incorporate *parity information* to yield invariants that are generally stronger than *biquandle brackets* when applied to virtual knots.

Recall that a *classical knot* is a knot in $\mathbb{R}^3$ and a *virtual knot* is a knot in a thickned orientable surface (note, all classical knots are virtual knots). An interesting property of virtual knots is that unlike classical knots, some of them can have crossings with non-zero *parity*:

Definition 4 (Parity)**.** Let K be a knot represented by some diagram D. For each crossing c in D, count the number of crossings encountered in traveling from the *overstrand* at c to the *understrand* at c; denote this quantity by n_c. Then we define the *parity* of c to be $n_c \bmod 2$.

We will incorporate parity in two places to enhance biquandle brackets. First, we will replace *biquandles* with *parity biquandles* (first introduced in [1]), then we will replace the biquandle bracket coefficient maps with parity-dependent versions.

Definition 5 (Parity Biquandle)**.** A *parity biquandle* is a set X together with four binary operations $\underline{\triangleright}^0, \overline{\triangleright}^0, \underline{\triangleright}^1, \overline{\triangleright}^1$ such that

(i) $(X, \underline{\triangleright}^0, \overline{\triangleright}^0)$ is a biquandle (note, $(X, \underline{\triangleright}^1, \overline{\triangleright}^1)$ need not be)

(ii) For all $x, y \in X$, the maps $x \overset{\alpha_y^1}{\longmapsto} x \overline{\triangleright}^1 y$, $x \overset{\beta_y^1}{\longmapsto} x \underline{\triangleright}^1 y$, and $(x, y) \overset{S}{\longmapsto} (y \overline{\triangleright}^1 x, x \underline{\triangleright}^1 y)$ are all invertible, and

(iii) For all $(a, b, c) \in \{(1,1,0), (1,0,1), (0,1,1)\}$ and for all $x, y, z \in X$, we have the *mixed exchange laws*

$$(z \overline{\triangleright}^a y) \overline{\triangleright}^b (x \overline{\triangleright}^c y) = (z \overline{\triangleright}^b x) \overline{\triangleright}^a (y \underline{\triangleright}^c x)$$
$$(x \overline{\triangleright}^a y) \underline{\triangleright}^b (z \overline{\triangleright}^c y) = (x \underline{\triangleright}^b z) \overline{\triangleright}^a (y \underline{\triangleright}^c z)$$
$$(y \underline{\triangleright}^a x) \underline{\triangleright}^b (z \overline{\triangleright}^c x) = (y \underline{\triangleright}^b z) \underline{\triangleright}^a (x \underline{\triangleright}^c z)$$

Now, we define *Kaestner Brackets:*

Definition 6. Let $\mathcal{X} = (X, \underline{\triangleright}^0, \overline{\triangleright}^0, \underline{\triangleright}^1, \overline{\triangleright}^1)$ be a parity biquandle, and let R be a commutative ring with identity. Let $A_0, B_0, A_1, B_1 : X \times X \to R^\times$. Then the collection $(\mathcal{X}, A_0, B_0, A_1, B_1)$ is a *Kaestner bracket* iff it satisfies the following conditions:

(i) $((X, \underline{\triangleright}^0, \overline{\triangleright}^0), A_0, B_0)$ is a biquandle bracket,

(ii) $A_1, B_1 : X \times X \to R^\times$ are invertible,

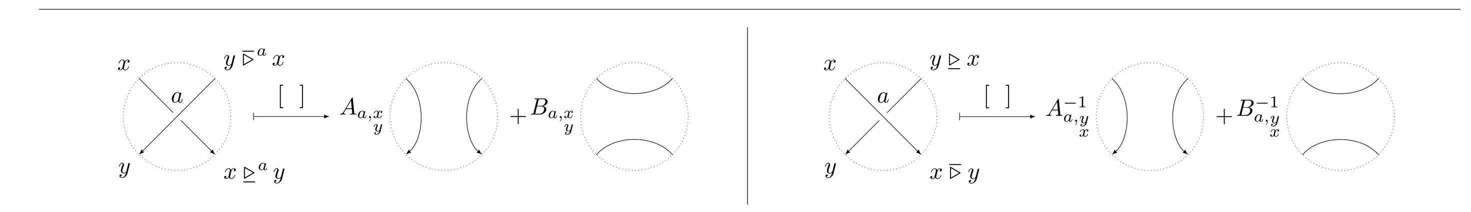
(iii) There exists some $\delta \in R$ such that for all $x, y \in X$,

$$-A_1(x,y) \cdot B_1^{-1}(x,y) - A_1^{-1}(x,y) \cdot B_1(x,y) = \delta$$

(iv) For all $x, y, z \in X$ and for all $(a, b, c) \in \{(1,1,0), (1,0,1), (0,1,1)\}$, we have the following (note, for the sake of notational compactness we write terms like $A_0(x, y)$ as $A_{0,\substack{x\\y}}$ instead):

$$A_{a,\substack{x\\y}} \cdot A_{b,\substack{x \underline{\triangleright}^a y\\ z \overline{\triangleright}^c y}} \cdot A_{c,\substack{y\\z}} = A_{c,\substack{y \overline{\triangleright}^a x\\ z \overline{\triangleright}^b x}} \cdot A_{b,\substack{x\\z}} \cdot A_{a,\substack{x \underline{\triangleright}^b z\\ y \underline{\triangleright}^c z}}$$
$$A_{a,\substack{x\\y}} \cdot B_{b,\substack{x \underline{\triangleright}^a y\\ z \overline{\triangleright}^c y}} \cdot B_{c,\substack{y\\z}} = B_{c,\substack{y \overline{\triangleright}^a x\\ z \overline{\triangleright}^b x}} \cdot B_{b,\substack{x\\z}} \cdot A_{a,\substack{x \underline{\triangleright}^b z\\ y \underline{\triangleright}^c z}}$$
$$B_{a,\substack{x\\y}} \cdot B_{b,\substack{x \underline{\triangleright}^a y\\ z \overline{\triangleright}^c y}} \cdot A_{c,\substack{y\\z}} = A_{c,\substack{y \overline{\triangleright}^a x\\ z \overline{\triangleright}^b x}} \cdot B_{b,\substack{x\\z}} \cdot B_{a,\substack{x \underline{\triangleright}^b z\\ y \underline{\triangleright}^c z}}$$
$$A_{a,\substack{x\\y}} \cdot B_{b,\substack{x \underline{\triangleright}^a y\\ z \overline{\triangleright}^c y}} \cdot A_{c,\substack{y\\z}} = A_{c,\substack{y \overline{\triangleright}^a x\\ z \overline{\triangleright}^b x}} \cdot A_{b,\substack{x\\z}} \cdot B_{a,\substack{x \underline{\triangleright}^b z\\ y \underline{\triangleright}^c z}} \quad B_{c,\substack{y \overline{\triangleright}^a x\\ z \overline{\triangleright}^b x}} \cdot A_{b,\substack{x\\z}} \cdot A_{a,\substack{x \underline{\triangleright}^b z\\ y \underline{\triangleright}^c z}}$$
$$+ \delta B_{c,\substack{y \overline{\triangleright}^a x\\ z \overline{\triangleright}^b x}} \cdot A_{b,\substack{x\\z}} \cdot B_{a,\substack{x \underline{\triangleright}^b z\\ y \underline{\triangleright}^c z}} + B_{c,\substack{y \overline{\triangleright}^a x\\ z \overline{\triangleright}^b x}} \cdot B_{b,\substack{x\\z}} \cdot B_{a,\substack{x \underline{\triangleright}^b z\\ y \underline{\triangleright}^c z}}$$
$$A_{c,\substack{y \overline{\triangleright}^a x\\ z \overline{\triangleright}^b x}} \cdot B_{b,\substack{x\\z}} \cdot A_{a,\substack{x \underline{\triangleright}^b z\\ y \underline{\triangleright}^c z}} = A_{a,\substack{x\\y}} \cdot A_{b,\substack{x \underline{\triangleright}^a y\\ z \overline{\triangleright}^c y}} \cdot B_{c,\substack{y\\z}} + B_{a,\substack{x\\y}} \cdot A_{b,\substack{x \underline{\triangleright}^a y\\ z \overline{\triangleright}^c y}} \cdot A_{c,\substack{y\\z}}$$
$$+ \delta B_{a,\substack{x\\y}} \cdot A_{b,\substack{x \underline{\triangleright}^a y\\ z \overline{\triangleright}^c y}} \cdot B_{c,\substack{y\\z}} + B_{a,\substack{x\\y}} \cdot B_{b,\substack{x \underline{\triangleright}^a y\\ z \overline{\triangleright}^c y}} \cdot B_{c,\substack{y\\z}}$$

These axioms guarantee that polynomials computed using the coloring-dependent skein relations below will be invariants of oriented links. For further details on the construction, see [2].



We can think of Kaestner brackets as a general cookbook for constructing knot polynomials with coloring-dependent skein relations. However, one should note that if we employ a *constant coloring* (i.e., all arcs are labeled identically), then we can also recover well-known invariants such as the *Jones*, *Alexander*, and *HOMFLYPT* polynomials as special cases of Kaestner brackets.

Results

- We have demonstrated examples of Kaestner brackets that outperform their classical biquandle bracket counterparts, thus the addition of parity information is meaningful!
- In pursuing computational results we made significant improvements on biquandle enumeration algorithms. In one case, we lowered runtime for a calculation from > 50 days to just 46 seconds.
- We also reformulated the axioms for *Trace Diagrams* (a digraph-based encoding scheme for knots) to yield representations that are more elegant theoretically and computationally.

Questions for Further Research

- Is there a way to relax the condition that δ be constant? In particular, can we make δ depend on the traces in our Trace Diagrams?
- As we've seen, including parity information improves the performance of our invariants in distinguishing virtual knots. Can we apply similar ideas by using remainder mod 4 to strengthen invariants of classical knots?
- For computational purposes, we restricted all of our searches to finite parity biquandles with bracket maps over finite fields. How can we extend these techniques to search for Kaestner brackets in infinite rings (e.g., $\mathbb{R}[x]$)?

Acknowledgments

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