Problems	4.31(F)	4.41	5.1	5.5	5.9	Total
Points						

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Math 147
HW 4 Solutions
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4.31(F) Consider the following subspace of the lexicographically ordered square:

$$F = \{(x,1) \mid 0 < x < 1\}.$$

As a set, it is a line. Describe its relative topology, noting any connections to the topologies you have seen already.

General Note: Honestly, the edge cases here make this problem a bit of a mess. Feel free to gloss over them by drawing diagrams for each case instead of actually writing things out formally. I've included everything here for the sake of completeness, but you should feel free to omit some parts. Just make it clear to me that you know *why* everything has the form it should! :)

Notational Notes: For the purposes of this problem, we will use boldface/angled brackets to denote points, and parentheses for open intervals. E.g., $\mathbf{p} = \langle a, b\rangle \in \mathbb{R}^2$, whereas $(a,b) \subset \mathbb{R}$. Note, under this notational scheme, "open intervals" in the lexicographically ordered square will thus be denoted by something like $(\langle a_0, b_0\rangle, \langle a_1, b_1\rangle)$:

$$\Big(\langle a_0, b_0\rangle, \langle a_1, b_1\rangle\Big) = \Big\{\langle x, y\rangle \;\Big|\; \langle a_0, b_0\rangle < \langle x, y\rangle < \langle a_1, b_1\rangle\Big\}$$

using square brackets when appropriate. Note that if we have previously defined $\mathbf{p}_0 = \langle a_0, b_0\rangle$, $\mathbf{p}_1 = \langle a_1, b_1\rangle$, then we would just write this as $(\mathbf{p}_0, \mathbf{p}_1)$.

Also, injection will be denoted $f : A \hookrightarrow B$, surjection by $f : A \twoheadrightarrow B$, and bijection by $f : A \hookrightarrow\!\!\!\!\!\rightarrow B$.

Solution. Let $(X_{\text{sq}}, \mathcal{T}_{\text{sq}})$ be the lexicographically ordered square, and $\mathcal{B}_{\text{sq}}$ be the canonical basis for $\mathcal{T}_{\text{sq}}$. Let $\mathcal{T}_F$ be the relative topology on F inherited from X_{sq}. Also let $(I, \mathcal{T}_I)$ be the interval $(0,1)$ together with the subspace topology inherited from $\mathbb{R}_{\text{LL}}$. Then we claim $(F, \mathcal{T}_F)$ is "equivalent" to $(I, \mathcal{T}_I)$ (denoted $\mathcal{T}_F \cong \mathcal{T}_I$).[1]

Proof: By theorem 4.30,

$$\mathcal{B}_F = \{B \cap F \mid B \in \mathcal{B}_{\text{sq}}\}$$

is a basis for $\mathcal{T}_F$. Thus, to show $\mathcal{T}_F \cong \mathcal{T}_I$, it will suffice to characterize elements of $\mathcal{B}_F$ (Claim 1) and put them in correspondence with a basis for $\mathcal{T}_I$ (Claim 2).

Claim 1: Elements of $\mathcal{B}_F$ are of the following forms:[2]

(1) $\varnothing$

(2) $\{\langle x, 1\rangle \in F \mid a < x < b\}$ (where $a, b \in (0,1)$)

(3) $\{\langle x, 1\rangle \in F \mid a \le x < b\}$ (where $a, b \in (0,1)$)

Proof of Claim 1: Let $B_F \in \mathcal{B}_F$ be arbitrary. Then $\exists B_{\text{sq}} \in \mathcal{B}_{\text{sq}}$ such that $B_F = B_{\text{sq}} \cap F$. Let $\mathbf{0} = \langle 0, 0\rangle$ and $\mathbf{1} = \langle 1, 1\rangle$. By definition of $\mathcal{B}_{\text{sq}}$, B_{sq} is of one of the following forms:

[1] Note: Technically, the formal term to use here is "homeomorphic." Homeomorphism is the notion of a topological "equivalence," or in the language of Category Theory, "isomorphism in the category of topological spaces". This won't be covered formally until section 8.3, but the idea is that homeomorphisms are bijections preserving open sets. This property of "preserving open sets" will become our topological definition of continuity, and in this context, a homeomorphism will be a continuous function with a continuous inverse.

[2] Note, (2) and (3) can also be written as $(a,b) \times \{1\}$ and $[a,b) \times \{1\}$, respectively

(a) $\Big[\mathbf{0}, \langle a_1, b_1\rangle\Big)$,

(b) $\Big(\langle a_0, b_0\rangle, \langle a_1, b_1\rangle\Big)$,

(c) $\Big(\langle a_0, b_0\rangle, \mathbf{1}\Big]$

We proceed by casework.

(a) Suppose $B_{\text{sq}} = \Big[\mathbf{0}, \langle a_1, b_1\rangle\Big)$. Then

$$B_{\text{sq}} \cap F = \Big\{\langle x, 1\rangle \in X_{\text{sq}} \;\Big|\; \mathbf{0} < \langle x, 1\rangle < \langle a_1, b_1\rangle\Big\}.$$

We will consider the cases of $\langle a_1, b_1\rangle \leq \langle 0, 1\rangle$ and $\langle 0, 1\rangle < \langle a_1, b_1\rangle$. Note that these cases are disjoint and exhaustive.

i) If $\langle a_1, b_1\rangle \leq \langle 0, 1\rangle$, $F \cap B = \varnothing$ (see diagram). This is of form (1).[3]

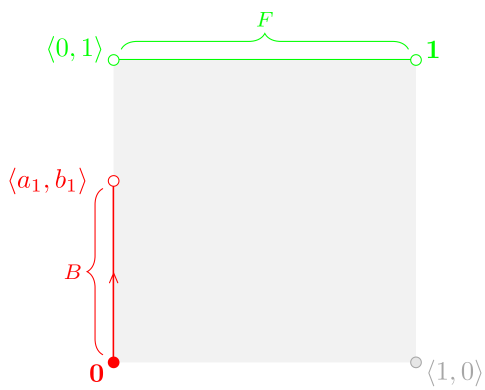


Figure 1: $F \cap B = \varnothing$.

ii) Suppose $\langle 0, 1\rangle < \langle a_1, b_1\rangle$. Then $F \cap B = \{\langle x, 1\rangle \in X_{\text{sq}} \mid x \in (0, a_1)\} = (0, a_1) \times \{1\}$ (see diagram).[4] This is of form (2).

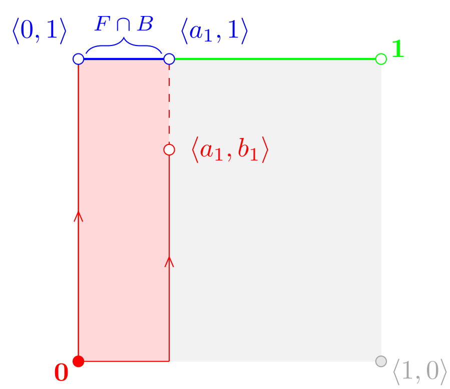


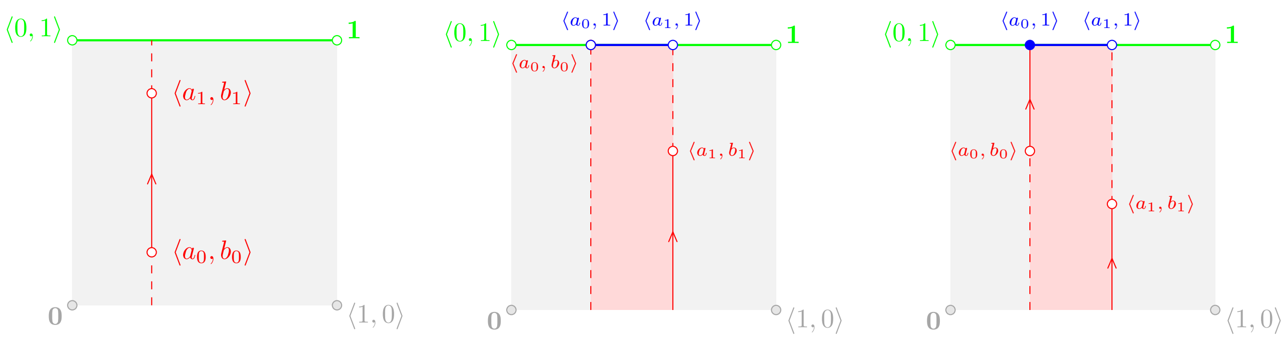
Figure 2: $F \cap B = (0, a_1) \times \{1\}$.

[3] Note this holds even if $\langle a_1, b_1\rangle = \langle 0, 1\rangle$, because the green endpoint is non-inclusive.

[4] Note that $\langle 0, 1\rangle \notin F \cap B$ by definition of F.

hence, if B_{sq} is of form (a), then $B_{\text{sq}} \cap F$ is of the desired form. ✓

(b) Now, suppose $B_{\text{sq}} = (\langle a_0, b_0\rangle, \langle a_1, b_1\rangle)$. Omitting the details, we have the following cases: (i) $a_0 = a_1$, (ii) $a_0 \neq a_1$ and $b_0 = 1$, and (iii) $a_0 \neq a_1$ and $b_0 \neq 1$.

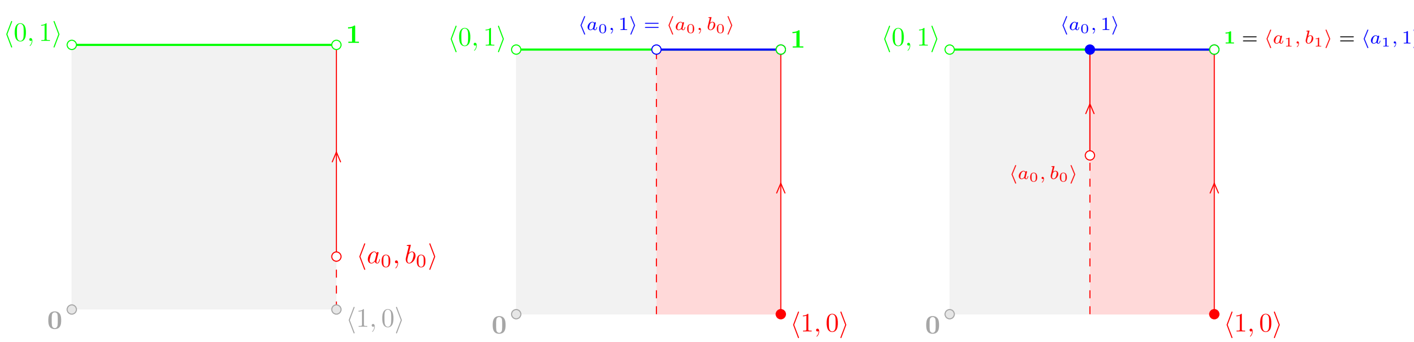


(i) `if` $a_0 = a_1$, $F \cap B = \varnothing$. (ii) `elif` $b_0 = 1$, $F \cap B = (a_0, a_1) \times \{1\}$ (iii) `else`, $F \cap B = [a_0, a_1) \times \{1\}$

Figure 3: Subcases

these have the forms (1), (2), and (3), respectively.

(c) Finally, suppose B_{sq} has the form $\Big(\langle a_0, b_0\rangle, \mathbf{1}\Big]$. Then we have the following cases: (i) $a_0 = 1$, (ii) $a_0 < 1$ and $b_0 = 1$, and (iii) $a_0 < 1$ and $b_0 < 1$. Note that these cases are disjoint and exhaustive.



(i) `if` $a_0 = 1$, $F \cap B = \varnothing$. (ii) `elif` $b_0 = 1$, $F \cap B = (a_0, 1) \times \{1\}$ (iii) `else`, $F \cap B = [a_0, 1) \times \{1\}$

Figure 4: Yet more subcases

which are of the forms (1), (2), and (3), respectively.

Since cases (a) – (c) are exhaustive, This proves the claim. (*Phew... finally!*)

> Note: you can stop here if you're doing your rewrite, as long as you just make a brief note about the similarity of $\mathcal{B}_F$ to a basis for $\mathcal{T}_I$.

Claim 2: We can establish a natural bijection between $\mathcal{B}_F$ and a basis $\mathcal{B}_I$ for $\mathcal{T}_I$, and extend this to a function $f : F \hookrightarrow\!\!\!\rightarrow I$ that respects the topologies $\mathcal{T}_F, \mathcal{T}_I$.

Proof of Claim 2: Let $\mathcal{B}_I$ be given by

$$\mathcal{B}_I = \{\varnothing\} \cup \Big\{(a, b) \subset (0, 1)\Big\} \cup \Big\{[a, b) \subset (0, 1)\Big\}.$$

Observe that $\mathcal{B}_I$ is indeed a basis for $\mathcal{T}_I$.[5] Thus, $f : \mathcal{B}_I \hookrightarrow\!\!\!\!\rightarrow \mathcal{B}_F$ defined by

$$f(B_I) = \begin{cases} \varnothing & \text{if } B_I = \varnothing \\ B_I \times \{1\} & \text{otherwise} \end{cases}$$

is a bijection with inverse given by

$$f^{-1}(B_F) = \pi_x(B_F) \qquad \text{(where } \pi_x \text{ is the canonical projection).}$$

We want to show $\forall U_I \in \mathcal{T}_I$, $f(U_I) \in \mathcal{T}_F$, and $\forall V_F \in \mathcal{T}_F$, $f^{-1}(V_F) \in \mathcal{T}_I$. Let $U_I \in \mathcal{T}_I$ be arbitrary. Then since $\mathcal{B}_I$ is a basis, there exists a collection of basis sets $\{B_{I,\alpha}\}_{\alpha\in\lambda} \subset \mathcal{B}_I$ such that

$$\bigcup_{\alpha\in\lambda} B_{I,\alpha} = U_I.$$

Hence

$$\begin{aligned} f(U_I) &= f\left(\bigcup_{\alpha\in\lambda} B_{I,\alpha}\right) && \text{(substituting for } U_I\text{)} \\ &= \bigcup_{\alpha\in\lambda} f(B_{I,\alpha}) && \text{(prop. of } \cup\text{)} \end{aligned}$$

For each $\alpha \in \lambda$, $f(B_{I,\alpha}) \in \mathcal{B}_F$ (by definition of f). Hence, $\{B_{F,\alpha}\}_{\alpha\in\lambda}$ defined by

$$B_{F,\alpha} = f(B_{I,\alpha}) \qquad (\alpha \in \lambda)$$

is a subset of $\mathcal{B}_F$. It follows that

$$f(U_I) = \bigcup_{\alpha\in\lambda} B_{F,\alpha}$$

is open in $(F, \mathcal{T}_F)$. The proof for arbitrary $V_F \in \mathcal{T}_F$ follows analogously.

Hence, f respects the topologies on F, I, so we have

$$(F, \mathcal{T}_F) \cong (I, \mathcal{T}_I)$$

as desired. ■

[5] In particular, the rightmost term is the collection of standard basis sets for $\mathcal{T}_I$, and the left two terms can be obtained by closure under arbitrary union.

4.41 Let $\mathbb{R}^\omega$ be the countable product of copies of $\mathbb{R}$. So every point in $\mathbb{R}^\omega$ is a sequence $x_1, x_2, x_3, \ldots$. Let $A \subset \mathbb{R}^\omega$ be the set consisting of all points with only positive coordinates. Show that in the product topology, $\mathbf{o} = (0, 0, 0, \ldots)$ is a limit point of the set A, and there is a sequence of points in A converging to $\mathbf{o}$. Then show that in the box topology, $\mathbf{o} = (0, 0, 0, \ldots)$ is a limit point of the set A, but there is no sequence of points in A converging to $\mathbf{o}$.

General Note: In this problem, we pick an arbitrary $U \in \mathcal{T}$. Lots of people assumed that U needs to be of the form

$$\prod_{i \in \mathbb{N}} U_i$$

where for each i, either $U_i = (a_i, b_i)$ or $U_i = \mathbb{R}$. This isn't how the box/product topologies were defined (we only know $U_i \in \mathcal{T}_{\text{std}}$, not $U_i \in \mathcal{B}_{\text{std}}$). However, I didn't take off points for this assumption, since this case captures the essential ideas of the proof.

Notational Note: Let A be a subset of a topological space. Then we will use $\mathcal{L}(A)$ to denote the limit points of A.

Solution. Let $\mathcal{T}_{\text{prod}}, \mathcal{T}_{\text{box}}$ be the product and box topologies on $\mathbb{R}^\omega$, respectively. Denote their corresponding bases by $\mathcal{B}_{\text{prod}}, \mathcal{B}_{\text{box}}$. We first show that in either topology $\mathbf{o} \in \mathcal{L}(A)$, and then show the results about sequences.

(1) WTS $\mathbf{o} \in \mathcal{L}(A)$. The following proof works in both $\mathcal{T}_{\text{prod}}$ and $\mathcal{T}_{\text{box}}$.[6] Hence, let $\mathcal{T}_\star$ refer to either one of them, and let $\mathcal{B}_\star$ be the corresponding basis.[7] We proceed by definition of a limit point.

Let $U \in \mathcal{T}_\star$ such that $\mathbf{o} \in U$. Then there exists $B \in \mathcal{B}_\star$ such that $B \subset U$, and $\mathbf{o} \in B$. By definition of $\mathcal{B}_\star$, B has the form

$$B = \prod_{i \in \mathbb{N}} V_i.$$

where for each i, V_i is open in $\mathbb{R}_{\text{std}}$.[8] $\mathbf{o} \in B$ implies that for each i, $0 \in V_i$, and thus there exists $(a_i, b_i) \subset \mathbb{R}$ with $0 \in (a_i, b_i) \subset V_i$. Taking

$$\mathbf{a} = \left(\frac{b_1}{2}, \frac{b_2}{2}, \ldots, \frac{b_i}{2}, \ldots\right)$$

we see that $\mathbf{a} \in B$ (and thus $\mathbf{a} \in U$), and $\mathbf{a} \in A$. Since $\mathbf{a} \neq \mathbf{o}$, we thus have

$$(U - \{\mathbf{o}\}) \cap A \neq \varnothing,$$

so $\mathbf{o} \in \mathcal{L}(A)$, as desired. ✓

(2) We want to show that under $\mathcal{T}_{\text{prod}}$, there exists $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \to \mathbf{o}$.

Claim: $\{x_n\}_{n \in \mathbb{N}}$ defined by

$$x_n = \left(\frac{1}{n}, \frac{1}{n}, \ldots\right)$$

works.

Proof of Claim: WTS $x_n \to \mathbf{o}$. Let $U \in \mathcal{T}_{\text{prod}}$ such that $\mathbf{o} \in U$. Then

$$U = \prod_{i \in \mathbb{N}} V_i$$

[6] This is because $\mathcal{T}_{\text{box}}$ is *finer* than $\mathcal{T}_{\text{prod}}$, and the proof works in $\mathcal{T}_{\text{box}}$.

[7] Formally, "let $\mathcal{T}_\star \in \left\{\mathcal{T}_{\text{prod}}, \mathcal{T}_{\text{box}}\right\}$ be arbitrary" — but this level of formality might distract the reader more than it helps, so I omit it.

[8] In the product topology, we require all but finitely many of the V_i to be copies of $\mathbb{R}$. But note, $\mathbb{R}$ is open in $\mathbb{R}_{\text{std}}$, so this statement actually *includes* the case that we're working in the product topology.

where each V_i is open, and for all but finitely many i, $V_i = \mathbb{R}$. Let $J \subset \mathbb{N}$ index the V_i for which $V_i \neq \mathbb{R}$.

$\mathbf{o} \in U$ implies that for each $j \in J$, $0 \in V_j$, and hence there exists (a_j, b_j) such that $0 \in (a_j, b_j) \subset V_j$. Since J is finite,

$$\min_{j \in J} \{b_j\} > 0,$$

so $\exists N \in \mathbb{N}$ with

$$\frac{1}{N} < \min_{j \in J} \{b_j\}.$$

Thus, for all $n > N$ we see $\frac{1}{n} \in V_i$ for each i, and hence $x_n \in U$. Since U was arbitrarily chosen, it follows that $x_n \to \mathbf{o}$.

(3) We want to show that under $\mathcal{T}_{\text{box}}$, there exists no $\{x_i\}_{i \in \mathbb{N}}$ with $x_i \to \mathbf{o}$.

Suppose, to obtain a contradiction, that such a sequence exists. Let the term x_i be given by

$$x_i = \left(b_i^{(1)}, b_i^{(2)}, \ldots\right)$$

Then consider U defined by

$$U = \prod_{i \in \mathbb{N}} \left(-b_i^{(i)}, b_i^{(i)}\right).$$

Since $x_i \to \mathbf{o}$, $\exists N \in \mathbb{N}$ st for all $n > N$, $x_n \in U$. But note,

$$b_n^{(n)} \notin \left(-b_n^{(n)}, b_n^{(n)}\right)$$

hence $x_n \notin U$, a contradiction.

Thus no such sequence exists.

■

5.1 A space $(X, \mathcal{T})$ is T_1 if and only if every point in X is a closed set.

> **General note:** This proof *can* be done by limit points. The resulting proof is almost exactly as long. However, I prefer the proof below, since it's constructive.

Solution.

($\Rightarrow$) : Suppose X is T_1. Let $x \in X$ be arbitrary. X is T_1 implies for all $y \in X - \{x\}$, there exists $U_y \in \mathcal{T}$ such that $x \notin U_y$.[9] Hence

$$X - \{x\} = \bigcup_{y \in X - \{x\}} U_y,$$

is a union of open sets, so $X - \{x\}$ open, and thus $\{x\}$ is closed. Since x was arbitrarily chosen, it follows that every point is a closed set (as desired).

($\Leftarrow$) : Suppose that every $\{p\} \subset X$ is closed. Let $x, y \in X$ such that $x \neq y$. $\{x\}, \{y\}$ are closed implies $X - \{x\}$ and $X - \{y\}$ are open. Observe that $y \in X - \{x\}$ and $x \in X - \{y\}$, but $x \notin X - \{x\}$ and $y \notin X - \{y\}$. Thus X is T_1.

■

[9] We're implicitly using the fact that $y \neq x \iff y \in X - \{x\}$ here.

5.5 Show that $\mathbb{R}_{\text{LL}}$ is normal.

Solution. Let $\mathscr{B}_{\text{LL}}$ denote the usual basis for $\mathbb{R}_{\text{LL}}$, and let A, B be arbitrary disjoint closed sets.

Let $a \in A$ be arbitrary. Since $A \cap B = \varnothing$, $a \in \mathbb{R} - B$. Observe that $\mathbb{R} - B$ is open (B is closed), thus there exists $[x_a, y_a) \in \mathscr{B}_{\text{LL}}$ such that $a \in [x_a, y_a) \subset \mathbb{R} - B$. Then

$$[a, y_a) \subset [x_a, y_a) \subset \mathbb{R} - B$$

as well. Define U by

$$U = \bigcup_{a \in A} [a, y_a),$$

and observe U is open, with $A \subset U$. By analogous reasoning, define

$$V = \bigcup_{b \in B} [b, y_b)$$

and observe V is open, with $B \subset V$.

Claim: $U \cap V = \varnothing$.

Proof of Claim: Suppose, to obtain a contradiction, that $U \cap V \neq \varnothing$.

> Note: This step in the leftbar is optional; you can skip it in your writeup. I've just included it for completeness.
>
> Then
>
> $$\begin{aligned} U \cap V &= \left(\bigcup_{a \in A} [a, y_a)\right) \cap \left(\bigcup_{b \in B} [b, y_b)\right) \\ &= \bigcup_{\substack{a \in A \\ b \in B}} [a, y_a) \cap [b, y_b) \qquad (\star) \\ &\neq \varnothing \end{aligned}$$
>
> where $(\star)$ follows by the distributive laws.

Then $\exists a \in A, b \in B$ such that $[a, y_a) \cap [b, y_b) \neq \varnothing$.

> Note: This step should *definitely* not be included in your writeup; it's far too much detail. I'm just including it here to be very explicit so that you can see every step if that's helpful :)
>
> Note that
>
> $$\begin{aligned} [a, y_a) \cap [b, y_b) &= \left\{x \in \mathbb{R} \;\middle|\; a \leq x < y_a\right\} \cap \left\{x \in \mathbb{R} \;\middle|\; b \leq x < y_b\right\} \\ &= \left\{x \in \mathbb{R} \;\middle|\; (a \leq x, b \leq x) \text{ and } (x < y_a, x < y_b)\right\} \\ &= \left\{x \in \mathbb{R} \;\middle|\; \max\{a, b\} \leq x < \min\{y_a, y_b\}\right\} \\ &= \Big[\max\{a, b\},\ \min\{y_a, y_b\}\Big). \end{aligned}$$
>
> Importantly, notice $\max\{a, b\}$ is an element of both $[a, y_a)$ and $[b, y_b)$! Since $[a, y_a) \subset \mathbb{R} - B$ and $[b, y_b) \subset \mathbb{R} - A$, this will get us our contradiction.

WLOG, suppose $a < b$. Then $b \in [a, y_a) \subset \mathbb{R} - B$, a contradiction ($b \notin \mathbb{R} - B$). Thus $U \cap V = \varnothing$, and this proves the claim. ✓

Since A, B were arbitrarily chosen, it follows that $\mathbb{R}_{\text{LL}}$ is normal. ■

5.9 A topological space X is normal if and only if for each closed set A in X and open set U containing A there exists an open set V such that $A \subset V$, and $\overline{V} \subset U$.

Solution.

($\Rightarrow$) : Suppose X is normal. Let A be an arbitrary closed set, and let $U \in \mathscr{T}$ such that $A \subset U$. Since U is open, $X - U$ is closed.

Further, note that $A \subset U$ implies $A \cap (X - U) = \varnothing$. Thus A, $X - U$ are disjoint closed sets. Then since X is normal, there exist disjoint open sets V, W such that $A \subset V$ and $(X - U) \subset W$.

Since V, W are disjoint, we have $V \subset X - W$. Furthermore, W is open implies $X - W$ is closed. Thus

$$\overline{V} \subset \overline{X - W} = X - W. \tag{1}$$

Now, note $(X - U) \subset W$ implies $(X - W) \subset U$. Thus, combining with (1), we see

$$A \subset V \subset \overline{V} \subset U$$

as desired.

($\Leftarrow$) : Suppose that for each $A \in \mathcal{C}$ and $U \in \mathscr{T}$ such that $A \subset U$, there exists $V \in \mathscr{T}$ such that $A \subset V$ and $\overline{V} \subset U$.

Let A, B be arbitrary disjoint closed sets. Then $A \subset X - B$, and $X - B$ is open. Thus by hypothesis, there exists open U such that

$$A \subset U \subset \overline{U} \subset X - B.$$

Note that $X - \overline{U}$ is open, and

$$B \subset X - \overline{U}.^{10}$$

Then by hypothesis, there exists open V such that

$$B \subset V \subset \overline{V} \subset X - \overline{U}$$

hence $\overline{U}, \overline{V}$ are disjoint, so U, V are disjoint as well. In the interests of being explicit, note that $A \subset U$ and $B \subset V$. Then since A, B were chosen arbitrarily, X is normal, as desired.

■

[10] In general, for sets C, D, $C \subset D$ implies $D^c \subset C^c$. Here, we've taken $C = \overline{U}$ and $D = X - B$.