Representation Theory Notes

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Contents

Introduction

What's this?

This document is a compendium of notes taken during Math 174 (Representation Theory) during Spring 2019. The book we used was *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions* by Bruce Sagan.

Notation

- "WTS" stands for "want to show," s.t. for "such that." WLOG, as usual, is without loss of generality.
- End-of-proof things: ■ is QED for exercises and theorems. □ is used in recursive proofs (e.g., proving a Lemma within a theorem proof). If doing a proof with casework, ✓ will be used to denote the end of each case.
- ($\Rightarrow\Leftarrow$) means contradiction
- $\mathcal{P}(A)$ is the powerset of A. I don't like using 2^A.
- $\twoheadrightarrow$ denotes surjection.
- $\hookrightarrow$ denotes injection.
- Thus, $\hookrightarrow\!\!\!\!\!\rightarrow$ denotes bijection.
- ϵ denotes trivial elements.
- $[n]$ will denote $\{1, \ldots, n\}$.
- Put some algebra things here at some point

1. Intro to Representation Theory (1/23/2019)

1.1 Some questions about edification

Discussion questions:

- What are the goals of a liberal arts education?
- How do we learn?
- What will you remember in 20 years after taking this course?
- What is the value of making mistakes in the learning process?
- How do we make a safe learning environment?

Students in the class had some pretty insightful responses at times.

1.2 Symmetry

Motivating question: What is Symmetry?

We have a lot of examples: in Physics, it can mean when we transform a system in some manner, but it remains invariant in some respect (example: Supersymmetry, something about exchanging bosons and fermions and stuff sort of works out the same). Thinking about a more everyday sense of the word, we might look at our right and left hands, and note that there's some biological symmetry here (they're chiral). Along these lines, we can look at symmetries of objects like squares and things, and quickly, we arrive at the dihedral group that we know and love.

So we might think of symmetry in terms of things like rotational symmetries, reflective symmetries, and so on. This points us in the direction of linear transformations. But we could also imagine translational symmetries, or symmetries that involve stretching our space along just one axis. So maybe we actually want affine transformations, or something of that form. This leads us to the "big-picture idea" in Rep. Theory:

Idea: We may think of symmetries as isomorphisms of vector spaces. In Rep. Theory, we try to take an arbitrary group $(G, \cdot)$, and try to reason about it as a group of *symmetries*, by finding a homomorphism $\varphi : G \to \text{Aut}(\mathcal{V})$, where $\mathcal{V}$ is a vector space.

1.3 The Symmetric Group

One might recall from Algebra I that every group is isomorphic to a subgroup of the symmetric group. Thus, maybe the statement above shouldn't surprise us *too* much — after all, the symmetric groups encode information about, well, symmetries of rigid objects, so we'd expect there to be ways of representing its subgroups in terms of "symmetries." Let's do some review.

Recall: The symmetric group S_n is the set of bijections from $\{1, \ldots, n\}$ to itself as a group under composition. Elements are called permutations, and every permutation admits a cycle decomposition.
We have a variety of notations by which to refer to elements of S_n. We'll use an example to illustrate them.

Ex: Let $\sigma \in S_5$ be given by $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 4$, $\sigma(4) = 2$, and $\sigma(5) = 5$. We can represent this diagramatically, or with permutation arrays / cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix} \quad \text{or} \quad (2\ 3\ 4)$$

Definition 1.3.1 (Cycle Type)**.** The cycle type of a partition $\sigma \in S_n$ is of the form $(1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ where m_i is the number of cycles of length i in a cycle decomposition of σ. For the cycle given in the example above, the cycle type is $(1^2, 2^0, 3^1, 4^0, 5^1)$.

Definition 1.3.2 (Involution)**.** An *involution* is a permutation $\sigma \in S_n$ such that $\sigma^2 = (1) = \epsilon$

Claim: σ is an involution iff $m_i = 0$ whenever $i > 2$.

Proof sketch.

($\Rightarrow$) : Suppose $m_i = 0$ for all $i > 2$. Then we can represent σ by $(i_1\ i_2) \cdots (i_{k-1}\ i_k)$. Because disjoint cycles commute and every 2-cycle is its own inverse, then we get $\sigma^2 = \epsilon$.

($\Leftarrow$) : Suppose $\sigma^2 = \epsilon$. Suppose, to obtain a contradiction, that σ contained a cycle of length $k > 2$ in its cycle decomposition. Since cycles of length k are order k, we quickly obtain a contradiction.

■

Recall: A *partition* of $n \in \mathbb{Z}$ is a monotonically decreasing sequence of integers $\lambda_1, \ldots, \lambda_k$ with $1 \leq k \leq n$ such that

$$\sum_{i=1}^{k} \lambda_i = n$$

oftentimes we will denote a partition by λ.

Observe that each partition of n determines (and is determined by) a cycle type in S_n. As an example, for $\sigma = (2\ 3\ 4)$, we have $\lambda = (3, 1, 1)$

Definition 1.3.3 (Conjugate elements)**.** In any group G, elements $g, h \in G$ are conjugate if $\exists x \in G$ s.t. $g = xhx^{-1}$.

Remark: Note that this is an equivalence relation. Let $\sim$ be the relation $g \sim h \iff g = xhx^{-1}$ for some $x \in G$.

Proof. Let G be a group, and let $\sim$ be defined as above. Let $g \in G$ be arbitrary. Taking $x = \epsilon$, we get $g = \epsilon g \epsilon^{-1}$, hence we have reflexivity. Now, let $h \in G$ be arbitrary. THen $g \sim h \iff \exists x \in G$ s.t. $g = xhx^{-1} \iff x^{-1}gx = h$, thus we have $h \sim g$, which gives us symmetry. Finally, we prove transitivity. Let $a \in G$, and suppose $g \sim a$ and $a \sim h$. Then $\exists x, y \in G$ s.t. $g = xax^{-1},\ a = yhy^{-1}$. Then $g = xyhy^{-1}x^{-1} = (xy)h(xy)^{-1}$, as desired. Thus, $\sim$ is an equivalence relation. ■

Corollary 1.3.1. *Wait, you can have corollaries to remarks? Oops... anyways, conjugacy classes partition the group, because of transitivity and stuff. For an element $g \in G$, we denote the conjugacy class of g by K_g.*

Recall: The centralizer of $g \in G$, denoted $\mathbf{Z}_g$ (and sometimes by $\mathbf{C}_G(g)$) is defined by $\mathbf{Z}_g = \left\{h \in G \;\middle|\; hgh^{-1} = g\right\} = \{h \in G \mid hg = gh\}$.

Theorem 1.3.2. *There is a bijection between the set of cosets of $\mathbf{Z}_g$ and K_g*

Proof of claim. Let $g \in G$. Let $x, y \in G$ such that $x\mathbf{Z}_g = y\mathbf{Z}_g$. Observe

$$\begin{aligned} x\mathbf{Z}_g = y\mathbf{Z}_g &\iff y^{-1}x\mathbf{Z}_g = \mathbf{Z}_g \\ &\iff y^{-1}x\mathbf{Z}_g \in \mathbf{Z}_g \\ &\iff y^{-1}xg = gy^{-1}x \\ &\iff y^{-1}xgx^{-1}y = g \\ &\iff xgx^{-1} \in K_g \quad \text{and} \quad ygy^{-1} \in K_g \end{aligned}$$

thus we see a natural correspondence between $\mathbf{Z}_g$ and K_g. ■

Corollary 1.3.3. *Let $g \in G$. Then $|G| = |K_g| \cdot |\mathbf{Z}_g|$.*

Proof. Since we proved the result above, we don't need to employ the Orbit-Stabilizer theorem to the conjugation action. Instead, we use Lagrange's Theorem. We have $|G| = |\mathbf{Z}_g| \cdot |G : \mathbf{Z}_g|$. But by the above, $|G : \mathbf{Z}_g| = |K_g|$, thus we obtain the desired result. ■

Theorem 1.3.4. *Let $\sigma, \tau \in S_n$. Then $\tau \in K_\sigma \iff \sigma$ and τ share the same cycle type.*

Proof. First, we prove a small Lemma.

Lemma 1.3.5. *Let $\sigma, \tau \in S_n$. Suppose $\tau = (i_1\ i_2\ \cdots\ i_\ell)\cdots(i_m\ i_{m+1}\ \cdots\ i_n)$. Then*

$$\sigma\tau\sigma^{-1} = (\sigma(i_1)\ \sigma(i_2)\ \cdots\ \sigma(i_\ell))\cdots(\sigma(i_m)\ \sigma(i_{m+1})\ \cdots\ \sigma(i_n)).$$

Proof of Lemma. Let $i_k \in [n]$. Then note that

$$\begin{aligned} (\sigma\tau\sigma^{-1})\sigma(i_k) &= (\sigma\tau\sigma^{-1}\sigma)(i_k) \\ &= (\sigma\tau)(i_k) \\ &= \begin{cases} \sigma(i_{k+1}) & \text{if } i_k \text{ is not one of the “ends” in a cycle in the decomposition} \\ \sigma(i_{k-j}) & \text{if it is,} \end{cases} \end{aligned}$$

thus we see $\sigma(i_k)$ maps to the place it should, per the claim. □

The claim now follows (relatively) quickly. ■

Notation: Recall two elements of S_n are conjugate iff they have the same cycle type. Hence, if $g \in S_n$ has cycle type λ, then we use the following notations:

$$K_g = k_\lambda, \qquad \mathbf{Z}_g = \mathbf{Z}_\lambda \qquad z_\lambda = |\mathbf{Z}_\lambda|$$

Theorem 1.3.6. *Let $g \in S_n$ have type $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$. Then $|\mathbf{Z}_g|$ depends only on λ. In particular,*

$$\mathbf{z}_g = \prod_{i=1}^{n} i^{m_i}(m_i!)$$

We'll leave this as a proposition for now.

2. Group Representations

2.1 Notes on the Reading (Sagan 1.1 — 1.4)

I'll skip some of the stuff that's redundant with respect to the material covered in class. That really leaves us with basically just one definition, namely the sign of a permutation.

Definition 2.1.1 ($\operatorname{sgn}(\sigma)$)**.** Call a 2-cycle a *transposition.* The transpositions generate S_n, in fact, the set of *adjacent* transpositions (i.e., $(1,2),\ (2,3),\ \ldots,\ (n-1,n)$ themselves generate the group of transpositions, hence they too generate S_n), hence we can decompose any cycle into a finite number of transpositions. For $\pi \in S_n$, let $\tau_1\tau_2\cdots\tau_k$ be such a decomposition. Then define the *sign* of π to be

$$\operatorname{sgn}(\pi) = (-1)^k.$$

sgn is well-defined, hence we can show that $\operatorname{sgn}(\pi\sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)$. This is our first example of a representation.

2.1.1 Matrix Representations (1.2)

We'll now return to our motivating question about symmetries. Matrix representations will provide the tool by which we can connect abstract groups to tangible symmetries of euclidean space. We'll begin with the definition, then give some examples.

Let $Mat_d(\mathbb{C})$ denote the set of all $d \times d$ matrices with entries in $\mathbb{C}$ (referred to as the *full complex matrix algebra of degree d*). Recall that algebras are vector spaces together with an associative product, thus imposing ring structure on the space. Taking the units under this multiplication operation and viewing them as a group under $\cdot$, we obtain $GL_d(\mathbb{C})$ the *complex general linear group of degree d.* Since we'll be dealing with $\mathbb{C}$ almost always, we'll simply shorten $Mat_d(\mathbb{C})$ and $GL_d(\mathbb{C})$ to Mat_d and GL_d, respectively.

Definition 2.1.2 (Matrix representation)**.** A *matrix representation* of a group G is a group homomorphism

$$X : G \to GL_d.$$

By the usual definition of a group homomorphism, this is equivalent to the requirements that $X(\epsilon) = I_d$, and $\forall g, h \in G,\ X(gh) = X(g)X(h)$. We call d the *degree* or *dimension* of the representation.

Example 2.1.1. Note that every group has the trivial representation by the trivial homomorphism. That is, $\forall g \in G,\ g \overset{X}{\longmapsto} (1) \in GL_1$.

Example 2.1.2. For $G = (\mathbb{Z}/n\mathbb{Z}, +)$, a one-dimensional representation of G is given by the n^{th} roots of unity, possibly under cyclic shift.

Definition 2.1.3 (Defining representation)**.** The defining representation of S_n is given by the homomorphism $X : S_n \to GL_n$ by

$$X(\pi) = \begin{cases} 1 & \text{if } \pi(j) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Note that this yields the set of permutation matrices!

2.1.2 G-Modules and the Group Algebra (1.3)

Important Note: Unless otherwise specified, all vector spaces are taken to be finite-dimensional spaces over $\mathbb{C}$.

Recall that matrices simply *encode* linear transformations, but linear transformations need not necessarily be encoded with respect to a particular basis (i.e., linear transformations aren't just matrices). We use this to construct G-modules, which will act like group actions but with the added requirement of linearity.

Let V be a vector space, and let $GL(V)$ be the set of all invertible linear transformations of V to itself, called the *general linear group* of V. If $\dim(V) = d$, then $GL(V) \cong GL_d$.

Definition 2.1.4 (G-module)**.** Let V be a vector space and G a group. Then V is a G-module if there is a group homomorphism

$$\rho : G \to GL(V).$$

In this case, we also say V *carries* a representation of G. If there can be no confusion about the group involved, we will often refer to a G-module simply as a module.

We also give an alternate definition in terms of the module axioms:

Definition 2.1.5 (G-module)**.** Let V be a vector space, and let G be a group. Then V is a G-module if we can define a multiplication $g\mathbf{v}$ (where $g \in G$ and $\mathbf{v} \in V$) such that

1. $g\mathbf{v} \in V$ (closure),
2. $g(c\mathbf{v} + d\mathbf{w}) = c(g\mathbf{v}) + d(g\mathbf{w})$ (linearity),
3. $(gh)\mathbf{v} = g(h\mathbf{v})$ (associativity),
4. $\epsilon\mathbf{v} = \mathbf{v}$

for all $g, h \in G$; $\mathbf{v}, \mathbf{w} \in V$; $c, d \in \mathbb{C}$.

The equivalence of the two definitions is given by recognizing that we take $g\mathbf{v}$ to mean $\rho(g)\mathbf{v}$. That is, g must act as a linear transformation. By existence of inverses (applied to axioms 3 and 4), we see that $\rho(g^{-1}) = (\rho(g))^{-1}$, thus our decision to restrict from Mat_d to GL_d is justified.

Definition 2.1.6 (Permutation Representation)**.** Let G be a group, and suppose G acts on a set S. Then the associated module $\mathbb{C}\mathbf{S}$ is called the *permutation representation* associated with S, and the elements of S form a basis for $\mathbb{C}\mathbf{S}$ called the *standard basis*.

Example 2.1.3. Let G be a group, and consider the action of G on itself by left multiplication. Extend the action to one on $\mathbb{C}[G]$ by linearity. Then the left regular representation of G is defined by taking the homomorphism $\rho : G \to GL(\mathbb{C}[G])$.

Definition 2.1.7 (Transversal)**.** Let G be a group with subgropu $H \leq G$. Then $g_1, g_2, \ldots g_k$ are a *transversal* for H iff $\mathcal{H} = \{g_1H, g_2H, \ldots, g_kH\}$ is a complete set of disjoint left cosets for H in G.

Example 2.1.4. Let G have a subgroup H, written $H \leq G$. A generalization of the regular representation is the (left) coset representation of G with respect to H. Let $g_1, g_2, \ldots, g_k$ be a transversal for H. Then G acts on $\mathcal{H}$ by

$$g(g_iH) = (gg_i)H$$

for all $g \in G$. The corresponding module

$$\mathbb{C}\mathcal{H} = \{c_1 g_1 \mathbf{H} + c_2 \mathbf{g}_2 \mathbf{H} + \cdots + c_k \mathbf{g}_k \mathbf{H} : c_i \in \mathbb{C} \forall i\}$$

inherits the action

$$g(c_1 \mathbf{g}_1 \mathbf{H} + \cdots + c_k \mathbf{g}_k \mathbf{H}) = (c_1 \mathbf{g}\mathbf{g}_1 \mathbf{H} + \cdots c_k \mathbf{g}\mathbf{g}_k \mathbf{H}).$$

If $H = G$, this is the trivial representation, while if $H = \{\epsilon\}$, we get the (left) regular representation.

2.1.3 Reducibility (1.4)

Definition 2.1.8 (Submodule)**.** Let V be a G-module. A *submodule* of V is a subspace W that is closed under the action of G, i.e.,

$$\mathbf{w} \in W \implies g\mathbf{w} \in W \ \forall g \in G.$$

We also say that W is a G-invariant subspace. Equivalently, W is a subset of V that is a G-module in its own right. We write $W \leq V$ if W is a submodule of V.

Example 2.1.5. Any G-module V has the submodules $W = V$, as well as $W = \{\mathbf{0}\}$ (where $\mathbf{0}$ is the zero vector). These are called *trivial*.

Example 2.1.6. Let $G = \mathcal{S}_n$. Then the sign representation can be recovered by using the submodule

$$W = \mathbb{C}\left[\sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi)\boldsymbol{\pi}\right].$$

Proof. ???? ∎

Definition 2.1.9 (Reducibility)**.** A nonzero G-module is *reducible* iff it contains a non-trivial submodule W. Otherwise, V is said to be *irreducible*. Equivalently, V is reducible iff it has a basis $\mathcal{B}$ in which every G is assigned a block matrix of the form

$$X(g) = \left[\begin{array}{c|c} A(g) & B(g) \\ \hline 0 & C(g) \end{array}\right]$$

where the $A(g)$ are square matrices (of the same size for all g), and 0 is a nonempty matrix of zeros.

Proof. We show the definitions are equivalent by showing each implies the other. First, suppose a nonzero G-module V contains a non-trivial submodule W. Pick a basis $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_f\}$ for W, where $0 < f < d$. Then let $\mathcal{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_f, \mathbf{v}_{f+1}, \ldots, \mathbf{v}_d\}$ be a basis for V. Then because W is a submodule of V, $\forall \mathbf{w}_i \in B, g \in G$, we have $g\mathbf{w}_i \in W$, that is

$$g\mathbf{w}_i = \sum_{i=1}^{f} c_i \mathbf{w}_i + \sum_{i=f+1}^{d} c_i \mathbf{v}_i$$

where $c_i = 0$ for $i = f+1, \ldots, d$. Then in this basis, we obtain the $A(g), 0$ blocks in the matrix representation of $X(g)$, and we see that $A(g)$ is $f \times f$, and 0 is $(d-f) \times f$. The $A(g)$ matrices all correspond to the restriction of the action of G to W, hence they must all be of the same size.

The converse direction is trivial. ∎

2.2 Complete Reducibility and Maschke's Theorem (Sagan 1.5)

We want to be able to write the matrices of a reducible G-module in block-diagonal form. Hence, we introduce the direct sum.

Definition 2.2.1. Let V be a vector space, and U, W subspaces of V. Then V is the (internal) direct sum of U and W, written $V = U \oplus W$ iff every $\mathbf{v} \in V$ can be written uniquely as

$$\mathbf{v} = \mathbf{u} + \mathbf{w}, \qquad \mathbf{u} \in U, \mathbf{w} \in W.$$

If V is a G-module and U, W are G-submodules, then we say that U and W are complements of each other.

Definition 2.2.2. If X is a matrix, then X is the *direct sum* of matrices A and B (written $X = A \oplus B$) if X has the block diagonal form

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

The reader should verify that the matrix representation of any $g \in G$ can be represented as

$$X(g) = \left(\begin{array}{c|c} A(g) & 0 \\ \hline 0 & B(g) \end{array}\right)$$

in some basis $\mathcal{B}$, where $A(g), B(g)$ are the matrices of the action of G restricted to U, W respectively.

Definition 2.2.3. Let $\mathbb{C}\mathbf{S}$ be a G-module. Given any two vectors $\mathbf{i}, \mathbf{j}$ in the basis $\mathcal{B}$, define the inner product on $\mathbb{C}\mathbf{S}$ by $\langle \mathbf{i}, \mathbf{j} \rangle = \delta_{i,j}$, and extension to linearity and the usual properties.

Definition 2.2.4. Let V be a vector space with inner product and subspace W. Then define the *orthogonal complement* of W by

$$W^\perp = \{\mathbf{v} \in V \mid \forall \mathbf{w} \in W, \langle \mathbf{v}, \mathbf{w} \rangle = 0\}.$$

Theorem 2.2.1. *Let V be a G-module, and W a submodule, and $\langle \cdot, \cdot \rangle$ an inner product invariant under the action of G. Then $W^\perp$ is a G-module.*

Proof. Let $\mathbf{u} \in W^\perp$. WTS $\forall g \in G, \forall \mathbf{w} \in W$, $\langle g\mathbf{u}, \mathbf{w} \rangle = 0$. Let $g \in G$, $w \in W$. be arbitrary, and let $\mathbf{w}' = g^{-1}\mathbf{w}$. Since W is a G-submodule, then $g^{-1}\mathbf{w} \in W$ as well. Because $\langle \cdot, \cdot \rangle$ is invariant under the action of G, we have

$$\begin{aligned} \langle g\mathbf{u}, \mathbf{w} \rangle &= \left\langle g^{-1} g\mathbf{u}, g^{-1}\mathbf{w} \right\rangle \\ &= \langle \mathbf{u}, \mathbf{w}' \rangle \\ &= 0 \end{aligned}$$

and so $W^\perp$ is closed under the action of G. ■

Theorem 2.2.2 (Maschke's Theorem)**.** *Let G be a finite group, and let V be a nonzero G-module. Then*

$$G = \bigoplus_{i=1}^{k} W^{(i)}$$

where each $W^{(i)}$ is an irreducible G-submodule of V.

Proof. We will induct on $d = \dim V$.

Base Case: Suppose $d = 1$. Then V is irreducible.

Inductive Hypothesis: Suppose the claim holds for $d = k$.

Inductive Step: Let V be a $d = k+1$-dimensional G-module. Suppose V is irreducible. Then we're done. Else, suppose that V is reducible. Then V contains a nontrivial sub-module W. Consider the inner product

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{i,j}.$$

Because $\langle \cdot, \cdot \rangle$ given above may not be G-invariant, we define an inner product that is.

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle' = \sum_{g \in G} \langle g\mathbf{v}, g\mathbf{w} \rangle.$$

The inner product axioms are satisfied by inheriting properties from $\langle \cdot, \cdot \rangle$. G-invariance is obtained by the fact that G acts on itself transitively. Thus, under $\langle \cdot, \cdot \rangle'$, we can decompose

$$V = W \oplus W^{\perp}$$

which, by hypothesis, both admit decompositions

$$W = \bigoplus_{i=1}^{j} W^{(i)} \qquad\qquad W^{\perp} = \bigoplus_{i=j+1}^{k+1} W^{(i)}$$

thus V can be represented as

$$V = \bigoplus_{i=1}^{k+1} W^{(i)}.$$

thus, by the Principle of Mathematical Induction, the claim holds for all d.

∎

Corollary 2.2.3. *Let G be a finite group, and let X be a matrix representation of G of dimension $d > 0$. Then $\exists T$ fixed such that $\forall g \in G$, $X(g)$ has the form*

$$TX(g)T^{-1} = \begin{pmatrix} X^{(1)}(g) & 0 & \cdots & 0 \\ 0 & X^{(2)}(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X^{(k)}(g) \end{pmatrix}$$

where each $X^{(i)}$ is an irreducible matrix representation of G restricted to $W^{(i)}$.

Proof. Simply make T the change of basis matrix mapping the standard basis of $\mathbb{C}^d$ to the basis $\mathcal{B}$ spanning the direct sum decomposition of W. ∎

Definition 2.2.5. A representation is *completely reducible* if it can be written as a direct sum of irreducibles. Thus, Maschke's theorem really states "Every representation of a finite group having positive dimension is completely reducible."

The finiteness condition is essential, while our choice of $\mathbb{C}$ as the underlying field is not.

2.3 G-Homomorphisms and Schur's Lemma (Sagan 1.6)

To study G-modules, we'll look at functions that preserve their structure.

Definition 2.3.1 (Module Homomorphism)**.** Let V, W be G-modules. Then a G-*homomorphism* is a linear transformation $\theta : V \to W$ such that

$$\theta(g\mathbf{v}) = g\theta(\mathbf{v})$$

for all $g \in G$ and $\mathbf{v} \in V$. We also say that θ *preserves* or *respects* the action of G.

Definition 2.3.2 (Basis-dependent definition)**.** Let V, W be G-modules. Suppose V basis $\mathcal{B}$, and W has basis $\mathcal{C}$. Let $X(g)$ and $Y(g)$ be the matrix representations corresponding to V and W, and let T be a $\dim(W) \times \dim(V)$ matrix. Then T is a matrix representation of a G-homomorphism iff $\forall g \in G$, $\mathbf{v} \in V$,

$$TX(g)\mathbf{v} = Y(g)T\mathbf{v}$$

whence

$$TX(g) = Y(g)T$$

Example 2.3.1. Let $G = \mathcal{S}_n$, and $V = \mathbb{C}\{\mathbf{v}\}$ with the trivial representation. Also let $W = \mathbb{C}\{\mathbf{1}, \mathbf{2}, \dots, \mathbf{n}\}$ with the defining representation of $\mathcal{S}_n$. Then $\theta : V \to W$ given by $\theta(\mathbf{v}) = \sum_{i=1}^{n} \mathbf{i}$ is a G-homomorphism.

Definition 2.3.3. Let V and W be modules for a group G. A G-*isomorphism* is a G-homomorphism $\theta : V \to W$ that is bijective.

Remark. This occurs whenever T is invertible. That is, whenever $\exists T$ s.t. $\forall g \in G$,

$$Y(g) = TX(g)T^{-1}$$

Definition 2.3.4. Let $\theta : V \to W$ be a vector space homomorphism. Then the *kernel* of θ, denoted $\ker(\theta)$, is given by

$$\ker(\theta) = \{\mathbf{v} \in V \mid \theta(\mathbf{v}) = \mathbf{0} \in W\}$$

and the *image* of θ, denoted $\operatorname{im}(\theta)$, is given by

$$\operatorname{im}(\theta) = \{\mathbf{w} \in W \mid \exists \mathbf{v} \in V \text{ s.t. } \theta(\mathbf{v}) = \mathbf{w}\}.$$

Proposition 2.3.1. *Let $\theta : V \to W$ be a G-homomorphism. Then*

(*a*) $\ker\theta$ *is a G-submodule of V, and*

(*b*) $\operatorname{im}\theta$ *is a G-submodule of W.*

Proof.

(a) Let $\mathbf{v} \in \ker(\theta)$. Then $\forall g \in G$, $\theta(g\mathbf{v}) = g\theta(\mathbf{v}) = g\mathbf{0} = \mathbf{0}$, hence $\ker\theta$ is closed under the action of G.

(b) Let $\mathbf{w} \in \operatorname{im}\theta$. Then $\exists \mathbf{v} \in V$ s.t. $\mathbf{w} = \theta(\mathbf{v})$. Then $g\mathbf{w} = \theta(g\mathbf{v}) \in \operatorname{im} W$, as desired.

■

Theorem 2.3.2 (Schur's Lemma)**.** *Let V and W be two irreducible G-modules. If $\theta : V \to W$ is a G-homomorphism, then either θ is a G-isomorphism, or θ is the zero map.*

Proof. $\ker(\theta)$ is a G-submodule of V. Since V is irreducible, we now have two cases:

(a) Suppose $\ker(\theta) = \{\mathbf{0}\}$. Then $\ker(\theta)$ is $1-1$. Now, since $\operatorname{im}(\theta)$ is a G-submodule of W, and W is irreducible, we have either $\operatorname{im}(\theta) = \{\mathbf{0}\}$ (in which case θ is the 0 map), or $\operatorname{im}(\theta) = W$, in which case θ is an isomorphism.

(b) Similar reasoning, but for $\ker(\theta) = V$.

■

Corollary 2.3.3. *Let X and Y be two irreducible matrix representations of G. If T is any matrix such that $TX(g) = Y(g)T$ for all $g \in G$, then either T is invertible, or $T = 0$.*

Corollary 2.3.4. *Let V and W be G-modules with V being irreducible. Then* $\dim \hom(V, W) = 0 \iff W$ *contains no submodule isomorphic to V.*

Corollary 2.3.5. *Let X be an irreducible matrix representation of G over $\mathbb{C}$. Then the only matrices T that commute with $X(g)$ for all $g \in G$ are scalar multiples of identity.*

2.4 Commutant and Endomorphism Algebras (Sagan 1.7)

The preceding corollaries indicate that it might be instructive to examine the set of matrices that commute with those of a given representation.

Definition 2.4.1. Given a matrix representation $X : G \to GL_d$, the corresponding *commutant algebra* is

$$\operatorname{Com}(X) = \{T \in \operatorname{Mat}_d \mid TX(g) = X(g)T \forall g \in G\},$$

where Mat_d is the set of all $d \times d$ matrices with entries in $\mathbb{C}$. Given a G-module V, the corresponding *endomorphism algebra* is

$$\operatorname{End}(V) = \{\theta : V \to V \mid \theta \text{ is a } G\text{-homomorphism}\}.$$

Example 2.4.1. Suppose that X is the matrix representation such that

$$X = \begin{pmatrix} X^{(1)} & 0 \\ 0 & X^{(2)} \end{pmatrix} = X^{(1)} \oplus X^{(2)},$$

where $X^{(1)}$ and $X^{(2)}$ are inequivalent with degrees d_1, d_2 respectively. We want to compute $\operatorname{End}(X)$. Let $T \in \operatorname{End}(X)$, partitioned in the same way as X:

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix}.$$

Then

$$TX = XT$$
$$\begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \begin{pmatrix} X^{(1)} & 0 \\ 0 & X^{(2)} \end{pmatrix} = \begin{pmatrix} X^{(1)} & 0 \\ 0 & X^{(2)} \end{pmatrix} \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix}$$

$$\begin{pmatrix} T_{1,1}X^{(1)} & T_{1,2}X^{(2)} \\ T_{2,1}X^{(1)} & T_{2,2}T^{(2)} \end{pmatrix} = \begin{pmatrix} X^{(1)}T_{1,1} & X^{(1)}T_{1,2} \\ X^{(2)}T_{2,1} & X^{(2)}T_{2,2} \end{pmatrix}.$$

By Schur's Lemma, $T_{1,2}X^{(2)} = X^{(1)}T_{1,2} \implies T_{1,2} = \mathbf{0}$ ($X^{(1)}$, $X^{(2)}$ are inequivalent). Similarly for $T_{2,1}$. Equating the remaining blocks, by (2.3.5), we have $T_{1,1}, T_{2,2}$ are multiples of identity matrices of degree d_1, d_2 respectively. Thus

$$T = \begin{pmatrix} c_1 I_{d_1} & \mathbf{0} \\ \mathbf{0} & c_2 I_{d_2} \end{pmatrix}.$$

So $\operatorname{Com} X = \{c_1 I_{d_1} \oplus c_2 I_{d_2} \mid c_1, c_2 \in \mathbb{C}\}$.

In fact, in general, when $X = \bigoplus_{i=1}^{k} X^{(i)}$ (pairwise inequivalent), we have

$$\operatorname{Com} X = \left\{ \bigoplus_{i=1}^{k} c_i I_{d_i} \;\middle|\; c_i \in \mathbb{C} \right\}$$

Definition 2.4.2. To prevent lots of errant $\oplus$'s flying around in the future, we'll employ the following notation:

$$mX = \overbrace{X \oplus X \oplus \cdots \oplus X}^{m \text{ times}}$$

where $m \in \mathbb{Z}^{\geq 0}$ is called the *multiplicity* of X.

Example 2.4.2. Suppose that $X = 2X^{(1)}$, where $X^{(1)}$ is irreducible of degree d. Then taking T partitioned as before, and performing analogous multiplications, we obtain

$$\operatorname{Com} X = \left\{ \begin{pmatrix} c_{1,1} I_d & c_{1,2} I_d \\ c_{2,1} I_d & c_{2,2} I_d \end{pmatrix} \;\middle|\; c_{i,j} \in \mathbb{C} \right\}$$

Definition 2.4.3. Let $X = (x_{i,j})$ and Y be matrices. Then their *tensor product* is the block matrix

$$X \otimes Y = (x_{i,j}Y) = \begin{pmatrix} x_{1,1}Y & x_{1,2}Y & \cdots \\ x_{2,1}Y & x_{2,2}Y & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Example 2.4.3. Let $X^{(1)}$ be an irreducible matrix representation, and let $m \in \mathbb{N}$. Then if X is a degree d representation of the form $X = mX^{(1)}$, we have

$$\operatorname{Com} X = \{M_m \otimes I_d \mid M_m \in \operatorname{Mat}_m\}$$

and we have $\deg(X) = md$, and $\dim(\operatorname{Com} X) = m^2$.

Example 2.4.4. Suppose X is of the form

$$X = m_1 X^{(1)} \oplus m_2 X^{(2)} \oplus \cdots \oplus m_k X^{(k)}$$

where each of the $m_i \in \mathbb{Z}^{>0}$, and $X^{(i)}$ is irreducible with degree d_i. Then

$$\operatorname{Com} X = \left\{ \bigoplus_{i=1}^{k} M_{m_i} \otimes I_{d_i} \;\middle|\; M_{m_i} \in \operatorname{Mat}_{m_i} \text{ for all } i \right\}$$

and

$$\dim(\operatorname{Com} X) = \sum_{i=1}^{k} m_i^2$$

We now define tensor products on abstract vector spaces:

Definition 2.4.4. Let V, W be vector spaces. Then their *tensor product* is given by the formal $\otimes$ products

$$V \otimes W = \left\{ \sum_{i,j} c_{i,j} \mathbf{v}_i \otimes \mathbf{w}_j \;\middle|\; c_{i,j} \in \mathbb{C},\ \mathbf{v}_i \in V,\ \mathbf{w}_j \in W \right\}$$

such that

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \otimes \mathbf{w} = c_1(\mathbf{v}_1 \otimes \mathbf{w}) + c_2(\mathbf{v}_2 \otimes \mathbf{w})$$

and

$$\mathbf{v} \otimes (d_1\mathbf{w}_1 + d_2\mathbf{w}_2) = d_1(\mathbf{v} \otimes \mathbf{w}_1) + d_2(\mathbf{v} \otimes \mathbf{w}_2)$$

Proposition 2.4.1. *If V and W are vector spaces, then $V \otimes W$ is also a vector space, and in fact, if $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$, $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_f\}$ are bases for V, W, then*

$$\{\mathbf{v}_i \otimes \mathbf{w}_j \mid 1 \leq i \leq d,\ 1 \leq j \leq f\}$$

is a basis for $V \otimes W$.

Proof. Let $c_1, c_2 \in \mathbb{C}$, and let $\mathbf{v}_1, \mathbf{v}_2 \in V$, $\mathbf{w}_1, \mathbf{w}_2 \in W$. Then

$$\begin{aligned} c_1(\mathbf{v}_1 \otimes \mathbf{w}_1) + c_2(\mathbf{v}_2 \otimes \mathbf{w}_2) &= c_1\left(\left[\sum_{i=1}^{d} a_{1,i}\mathbf{v}_i\right] \otimes \left[\sum_{j=1}^{f} b_{1,j}\mathbf{w}_j\right]\right) + c_2\left(\left[\sum_{i=1}^{d} a_{2,i}\mathbf{v}_i\right] \otimes \left[\sum_{j=1}^{f} b_{2,j}\mathbf{w}_j\right]\right) \\ &= c_1\left(\sum_{i=1}^{d}\sum_{j=1}^{f} a_{1,i}b_{1,j}\mathbf{v}_i \otimes \mathbf{w}_j\right) + c_2\left(\sum_{i=1}^{d}\sum_{j=1}^{f} a_{2,i}b_{2,j}\mathbf{v}_i \otimes \mathbf{w}_j\right) \\ &= \sum_{i,j} (c_1a_{1,i}b_{1,j} + c_2a_{2,i}b_{2,j})\mathbf{v}_i \otimes \mathbf{w}_j \end{aligned}$$

from which both results follow. ■

Remark. To make a connection with the definition of the matrix tensor product, observe that Mat_d has the basis

$$\mathcal{B} = \{E_{i,j} \mid q \leq i, j \leq d\}$$

where $E_{i,j} = 0$ except for a 1 the i, jth entry. From this, we can obtain the matrix definition of the tensor product.

Definition 2.4.5. The *center* of an algebra A is

$$\mathbf{Z}_A = \{a \in A \mid ab = ba \text{ for all } b \in A\}.$$

Proposition 2.4.2. *The* center *of* Mat_d *is*

$$\mathbf{Z}_{\mathrm{Mat}_d} = \{cI_d \mid c \in \mathbb{C}\}$$

Proof. Let $C \in \mathbf{Z}_{\mathrm{Mat}_d}$. Then $\forall E_{i,i}$,

$$CE_{i,i} = E_{i,i}C$$

$$\begin{pmatrix} 0 & \cdots & c_{1,i} & \cdots & 0 \\ 0 & \cdots & c_{2,i} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & c_{n,i} & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ c_{i,1} & c_{i,2} & \cdots & c_{i,n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

hence $c_{i,j} = 0$ for $i \neq j$. Similarly, we have

$$C(E_{i,j} + E_{j,i}) = (E_{i,j} + E_{j,i})C$$

$$\begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & c_{i,i} & \vdots \\ & c_{j,j} & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & c_{j,j} & \vdots \\ & c_{i,i} & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix}$$

hence $c_{i,i} = c_{j,j}$ for all i, j. So $C = cI$ for some $c \in \mathbb{C}$, as desired. ■

Lemma 2.4.3. *Suppose $A, X \in \mathrm{Mat}_d$ and $B, Y \in \mathrm{Mat}_f$. Then*

1. $(A \oplus B)(X \oplus Y) = AX \oplus BY$ *(block matrices of the same dimensions multiply by blocks)*
2. $(A \otimes B)(X \otimes Y) = AX \otimes BY$

Proof.

1. Recall that for $n \times n$ matrices M, N, matrix multiplication is defined by

$$MN = \left(\sum_{k=1}^{n} a_{i,k} b_{k,j} \right)_{i,j}.$$

   Note that $(A \oplus B)_{i,j} = 0$ whenever ($d < i$ and $j < f$) or ($i < d$ and $f < j$), and similarly with $X \oplus Y$. Applying this fact yields the desired form.

2. We have

$$\begin{aligned} \begin{pmatrix} a_{1,1}B & \cdots & a_{1,d}B \\ \vdots & \ddots & \vdots \\ a_{d,1}B & \cdots & a_{d,d}B \end{pmatrix} \begin{pmatrix} x_{1,1}Y & \cdots & x_{1,d}Y \\ \vdots & \ddots & \vdots \\ x_{d,1}Y & \cdots & x_{d,d}Y \end{pmatrix} &= \left(\sum_{k=1}^{d} a_{i,k} x_{k,j} BY \right)_{i,j} \\ &= \left(\sum_{k=1}^{d} a_{i,k} x_{k,j} \right)_{i,j} \otimes BY \\ &= AX \otimes BY \end{aligned}$$

   as desired.

■

Theorem 2.4.4. *Let X be a matrix representation of G such that*

$$X = m_1 X^{(1)} \oplus m_2 X^{(2)} \oplus \cdots \oplus m_k X^{(k)}$$

where the $X^{(i)}$ are inequivalent, irreducible and $\deg X^{(i)} = d_i$. Then

1. $\deg X = m_1 d_1 + m_2 d_2 + \cdots m_k d_k$,
2. $\mathrm{Com}\, X = \left\{ \bigoplus_{i=1}^{k} (M_{m_i} \otimes I_{d_i}) \;\middle|\; \forall i = 1, \ldots, k,\; M_{m_i} \in \mathrm{Mat}_{d_i} \right\}$

3. $\dim(\mathrm{Com}\, X) = m_1^2 + m_2^2 + \cdots + m_k^2$
4. $\mathbf{Z}_{\mathrm{Com}\, X} = \left\{ \bigoplus_{i=1}^k c_i I_{m_i d_i} \,\middle|\, \forall i = 1, \ldots, k,\ c_i \in \mathbb{C} \right\}$
5. $\dim \mathbf{Z}_{\mathrm{Com}\, X} = k$.

Proof. We've already proven the first three. Let $C \in \mathbf{Z}_{\mathrm{Com}\, X}$. Then $\forall T = \oplus_{i=1}^k (M_{m_i} \otimes I_{d_i})$,

$$
\begin{aligned}
CT &= TC \\
\left(\bigoplus_{i=1}^k C_{m_i} \otimes I_{d_i} \right) \left(\bigoplus_{i=1}^k M_{m_i} \otimes I_{d_i} \right) &= \left(\bigoplus_{i=1}^k M_{m_i} \otimes I_{d_i} \right) \left(\bigoplus_{i=1}^k C_{m_i} \otimes I_{d_i} \right) \\
\bigoplus_{i=1}^k (C_{m_i} \otimes I_{d_i})(M_{m_i} \otimes I_{d_i}) &= \bigoplus_{i=1}^k (M_{m_i} \otimes I_{d_i})(C_{m_i} \otimes I_{d_i}) \\
\bigoplus_{i=1}^k (C_{m_i} M_{m_i} \otimes I_{d_i}) &= \bigoplus_{i=1}^k (M_{m_i} C_{m_i} \otimes I_{d_i})
\end{aligned}
$$

thus $\forall i = 1, \ldots, k$, $C_{m_i} M_{m_i} = M_{m_i} C_{m_i}$. Since the M_{m_i} can be arbitrary, it follows that $C_{m_i} \in \mathbf{Z}_{\mathrm{Mat}_{m_i}}$, and hence $C_{m_i} = c_i I_{m_i}$. Thus,

$$
\begin{aligned}
C &= \bigoplus_{i=1}^k c_i I_{m_i} \otimes I_{d_i} \\
&= \bigoplus_{i=1}^k c_i I_{m_i d_i}
\end{aligned}
$$

as desired. Note that the $I_{m_i d_i}$ form a basis for $\mathbf{Z}_{\mathrm{Mat}_{m_i}}$, thus it has dimension k. ■

Theorem 2.4.5. *We restate the above in terms of modules. Let V be a G-module such that*

$$V \cong m_1 V^{(1)} \oplus m_2 V^{(2)} \oplus \cdots \oplus m_k V^{(k)},$$

where the $V^{(i)}$ are pairwise inequivalent irreducible and $\dim V^{(i)} = d_i$. Then

1. $\dim V = m_1 d_1 + m_2 d_2 + \cdots + m_k d_k$
2. $\mathrm{End}\, V \cong \bigoplus_{i=1}^k \mathrm{Mat}_{m_i}$
3. $\dim(\mathrm{End}\, V) = m_1^2 + m_2^2 + \cdots + m_k^2$
4. $\mathbf{Z}_{\mathrm{End}\, V} \cong \{ M \in \mathrm{Mat}_k \mid M \text{ is diagonal} \}$
5. $\dim \mathbf{Z}_{\mathrm{End}\, V} = k$.

Proposition 2.4.6. *Let V, W be G-modules with V irreducible. Then* $\dim \hom(V, W)$ *is the multiplicity of V in W.*

Proof. Sketch: we get one basis isomorphism $\theta^{(i)}$ for each of the copies $V^{(i)}$ of V in W. By Schur's Lemma, we get nothing else. ■

2.5 Group Characters (Sagan 1.8)

Definition 2.5.1 (Character)**.** Let $X(g), g \in G$, be a matrix representation. Then the *character* of X is the map $\chi : G \to \mathbb{C}$ defined by

$$\chi(g) = \operatorname{tr} X(g)$$

where tr is the trace.

We check that tr is well-defined. Let X, Y be two matrix representations corresponding to a G-module V. Then there exists a fixed matrix T such that $Y = TXT^{-1}$. Since trace is invariant over cyclic permutations,

$$\begin{aligned}\operatorname{tr} Y &= \operatorname{tr} TXT^{-1} \\ &= \operatorname{tr} T^{-1}TX \\ &= \operatorname{tr} X\end{aligned}$$

as desired.

Note. Much of the language we've developed so far for representations will carry over to characters. For example, we will say χ is irreducible whenever X is.

Example 2.5.1. For the defining representation of $\mathcal{S}_n$, the character $\chi^{\text{def}}(\pi)$ is given by the number of fixed points of π.

Example 2.5.2. For the regular representation of $V = \mathbb{C}[\mathbf{G}]$ and character χ^{reg},

$$\chi^{\text{reg}}(g) = \begin{cases} |G| & \text{if } g = \epsilon \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2.5.1. *Let X be a matrix representation of a group G of degree d with character χ. Then*

1. $\chi(\epsilon) = d$.
2. *If K is a conjugacy class of G, then*

$$g, h \in K \implies \chi(g) = \chi(h).$$

3. *If Y is a representation of G with character ψ, then*

$$X \cong Y \implies \chi(g) = \psi(g)$$

for all $g \in G$.

Proof. (a) $\operatorname{tr} X(\epsilon) = \operatorname{tr} I_d = d$.

(b) Let $g, h \in K$. By definition of a conjugacy class, there exists $x \in G$ such that $g = xhx^{-1}$. Then

$$\begin{aligned}\chi(g) &= \chi(xhx^{-1}) \\ &= \operatorname{tr}\left(X(x)X(h)X(x^{-1})\right) \\ &= \operatorname{tr}\left(X(x^{-1})X(x)X(h)\right) \\ &= \operatorname{tr}\left(X^{-1}(x)X(x)X(h)\right) \\ &= \operatorname{tr} X(h)\end{aligned}$$

as desired.

(c) See proof of well-defined-ness above.

■

Definition 2.5.2 (Class function)**.** A *class function* on a group G is a mapping $f : G \to \mathbb{C}$ such that $f(g) = f(h)$ whenever g and h are in the same conjugacy class. The set of all class functions on G is denored by $R(G)$.

Remark. $R(G)$ is a vector space over $\mathbb{C}$, with $\dim R(G)$ being the number of conjugacy classes in G.

Definition 2.5.3 (χ_K)**.** Since χ is invariant over a given conjugacy class, we can define χ_K by

$$\chi_K = \chi(g)$$

for some representative element $g \in K$.

Definition 2.5.4 (Character table)**.** Let G be a group. The *character table* of G is an array with rows indexed by the inequivalent irreducible characters of G and the columns indexed by the conjugacy classes. The $(\chi, K)^{\text{th}}$ entry of the table is χ_K.

2.6 Inner Products of Characters (Sagan 1.9)

We can consider the character of a group G to be a row vector in $\mathbb{C}^{|G|}$:

$$\chi = (\chi(g_1), \ldots, \chi(g_n)).$$

By computing some examples, we conjecture $\left\langle \chi^{(i)} \cdot \chi^{(j)} \right\rangle = |G| \cdot \delta_{i,j}$.

Definition 2.6.1 (Inner product of maps from G to $\mathbb{C}$)**.** Let $\chi, \psi : G \to \mathbb{C}$. Then

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}$$

Proposition 2.6.1. *Let χ, ψ be characters. Then*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1})$$

Proof. First, we show that there exists a basis in which Y is unitary. Recall the G-invariant inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle' = \sum_{g \in G} \langle g\mathbf{v}, g\mathbf{w} \rangle.$$

Let $\mathbf{e}_i, \mathbf{e}_j$ be basis elements of an orthonormal basis induced by $\langle \cdot, \cdot \rangle'$. Let $g_0 \in G$ be fixed. Then $Y(g_0)\mathbf{e}_i, Y(g_0)\mathbf{e}_j$ are an arbitrary pair of columns of Y.

$$\begin{aligned} \langle Y(g_0)\mathbf{e}_i, Y(g_0)\mathbf{e}_j \rangle' &= \frac{1}{|G|} \sum_{g \in G} \langle gY(g_0)\mathbf{e}_i, gY(g_0)\mathbf{e}_j \rangle \\ &= \frac{1}{|G|} \sum_{g' \in G} \langle g'\mathbf{e}_i, g'\mathbf{e}_j \rangle \\ &= \langle \mathbf{e}_i, \mathbf{e}_j \rangle' \\ &= \delta_{i,j} \end{aligned}$$

as desired. Hence, there exists a fixed change of basis matrix T such that

$$Y = TU(g)T^{-1}$$

where $U(g)$ is a unitary matrix. Hence, for all $g \in G$,

$$\begin{aligned}
\overline{\psi(g)} &= \overline{\operatorname{tr} Y(g)} \\
&= \operatorname{tr} \overline{TU(g)T^{-1}} \\
&= \operatorname{tr} \overline{U(g)T^{-1}T} \\
&= \operatorname{tr} \overline{U(g)} \\
&= \operatorname{tr} \overline{U^{\intercal}(g)} \\
&= \operatorname{tr} U^{-1}(g) \\
&= \operatorname{tr} U(g^{-1}) \\
&= \operatorname{tr} TU(g^{-1})T^{-1} \\
&= \operatorname{tr} Y(g^{-1}) \\
&= \psi(g^{-1})
\end{aligned}$$

as desired. ■

Remark. When χ, ψ are class functions (not just limited to characters), we have

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_K |K| \chi_K \overline{\psi_K},$$

Theorem 2.6.2 (Character Relations of the First Kind)**.** *Let χ, ψ be irreducible characters of a group G. Then*

$$\langle \chi, \psi \rangle = \delta_{\chi,\psi}.$$

Proof. We apply Schur's Lemma. Let A, B be the matrices corresponding to χ, ψ respectively, and let their degrees be d, f. Then we want to find a matrix $T : \mathbb{C}^f \to \mathbb{C}^d$ such that

$$A(g)T = TB(g)$$

for all $g \in G$. Consider

$$T = \sum_{g \in G} A(g)XB(g^{-1}).$$

where X is an arbitrary $d \times f$ matrix. Note, we only include X here so that the dimensions align. Observe that for all $h \in G$,

$$\begin{aligned}
A(h)TB(h^{-1}) &= \frac{1}{|G|} \sum_{g \in G} A(h)A(g)XB(g^{-1})B(h^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} A(hg)XB(g^{-1}h^{-1}) \\
&= \frac{1}{|G|} \sum_{\tilde{g} \in G} A(\tilde{g})XB(\tilde{g}^{-1}) \\
&= T
\end{aligned}$$

hence T is indeed the desired matrix. By Schur's Lemma,

$$T = \begin{cases} 0 & \text{if } A \ncong B, \\ cI & \text{if } A \cong B \end{cases}$$

where $c \in \mathbb{C}$, $I \in \operatorname{Mat}_d = \operatorname{Mat}_f$. We now have the following two cases:

(a) Suppose $A \not\cong B$. Then $T = 0$, hence

$$\begin{aligned}(T_{i,j}) &= \sum_{g \in G} (A(g)XB(g^{-1}))_{i,j} \\ &= \sum_{g \in G} \sum_{k,l} a_{i,k}(g) x_{k,l} b_{l,j}(g^{-1}) \\ &= 0\end{aligned}$$

since this is a polynomial in the $x_{k,l}$, for each of the terms to be 0, we must have

$$\begin{aligned}\sum_{g \in G} \sum_{k,l} a_{i,k}(g) b_{l,j}(g^{-1}) &= \langle a_{i,k}, b_{l,j} \rangle' \\ &= 0\end{aligned}$$

hence by linearity, $\langle \chi, \psi \rangle = \sum_{i,j} \langle a_{i,i}, b_{j,j} \rangle = 0$.

(b) Suppose $A \cong B$. Then

$$\begin{aligned}\operatorname{tr} T &= \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(A(g)XB(g^{-1})) \\ cd &= \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(A(g)XT^{-1}A(g^{-1})T) \\ &= \frac{1}{|G|} \sum_{g \in G} \operatorname{tr} X.\end{aligned}$$

Since $T = cI$, $T_{i,i} = c$, and so

$$\begin{aligned}T_{i,i} &= \frac{1}{d} \operatorname{tr} X \\ &= \frac{1}{d}(x_{1,1} + x_{2,2} + \cdots + x_{n,n}) \\ &= \frac{1}{d} \frac{1}{|G|} \sum_{g \in G} \sum_{k,l} a_{i,k}(g) x_{k,l} a_{l,i}(g^{-1})\end{aligned}$$

equating the terms of the polynomial between the two lines yields

$$\langle a_{i,k}, a_{l,i} \rangle' = \frac{1}{|G|} \sum_{g \in G} a_{i,k}(g) a_{l,i}(g^{-1}) = \frac{1}{d} \delta_{k,l}.$$

For $T_{i,j}$ with $i \neq j$, $y_{i,j} = 0$ forces

$$\langle a_{i,k}, a_{l,j} \rangle' = 0$$

for all k, l. Hence

$$\begin{aligned}\langle \chi, \psi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \operatorname{tr} A(g) \operatorname{tr} A(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{d} a_{i,i}(g) \sum_{j=1}^{d} a_{j,j}(g^{-1})\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|G|} \sum_{g \in G} \sum_{i,j=1}^{d} a_{i,i}(g) a_{j,j}(g^{-1}) \\
&= \sum_{i,j=1}^{d} \langle a_{i,i}, a_{j,j} \rangle' \\
&= \sum_{i=1}^{d} \langle a_{i,i}, a_{i,i} \rangle' \\
&= 1
\end{aligned}
$$

as desired.

■

Corollary 2.6.3. *Let X be a matrix representation of G with character χ. Suppose*

$$X \cong m_1 X^{(1)} \oplus m_2 X^{(2)} \oplus \cdots \oplus m_k X^{(k)}$$

where the $X^{(i)}$ are pairwise inequivalent irreps with characters $\chi^{(i)}$. Then

(1) $\chi = m_1 \chi^{(1)} + m_2 \chi^{(2)} + \cdots + m_k \chi^{(k)}$

(2) $\left\langle \chi, \chi^{(j)} \right\rangle = m_j$ *for all* j

(3) $\langle \chi, \chi \rangle = m_1^2 + m_2^2 + \cdots + m_k^2$

(4) X *is an irrep iff* $\langle \chi, \chi \rangle = 1$

(5) Let Y be another matrix rep of G with character ψ. Then

$$X \cong Y \iff \chi(g) = \psi(g) \qquad \forall g \in G$$

Proof. (1) Let $g \in G$. Then by properties of block matrices,

$$
\begin{aligned}
\chi &= \operatorname{tr} X \\
&= \operatorname{tr}\left(\bigoplus_{i=1}^{k} m_i X^{(i)} \right) \\
&= \sum_{i=1}^{k} m_i \operatorname{tr} X^{(i)} \\
&= \sum_{i=1}^{k} m_i \chi^{(i)}
\end{aligned}
$$

as desired.

(2) We have

$$
\begin{aligned}
\left\langle \chi, \chi^{(j)} \right\rangle &= \left\langle \sum_{i=1}^{k} m_i \chi^{(i)}, \chi^{(j)} \right\rangle \\
&= \sum_{i=1}^{k} m_i \left\langle \chi^{(i)}, \chi^{(j)} \right\rangle \\
&= \sum_{i=1}^{k} m_i \delta_{i,j} \\
&= m_j
\end{aligned}
$$

as desired.

(3) We have

$$\begin{aligned}\langle \chi, \chi \rangle &= \left\langle \sum_{i=1}^{k} m_i \chi^{(i)}, \sum_{i=1}^{k} m_i \chi^{(i)} \right\rangle \\ &= \sum_{i,j=1}^{k} m_i m_j \left\langle \chi^{(i)}, \chi^{(j)} \right\rangle \\ &= \sum_{i,j=1}^{k} m_i m_j \delta_{i,j} \\ &= \sum_{i=1}^{k} m_i^2\end{aligned}$$

as desired.

(4) Follows directly from (3). X is an irrep iff $\exists i_0$ such that

$$m_i = \delta_{i,i_0}$$

hence $\langle \chi, \chi \rangle = 1$. For the reverse direction, note that the m_i are all positive integers, hence $\sum_{i=1}^{d} m_i^2 = 1$ implies $\exists i_0$ s.t. $m_i = \delta_{i,i_0}$.

(5) $(\Rightarrow)$: Suppose $X \cong Y$. WTS $\chi = \psi$. We've proven this already, but as a reminder, it follows from $X = TYT^{-1}$ and cyclic permutation invariance of trace.

$(\Leftarrow)$: Suppose $\chi = \psi$. By Maschke's Theorem, X and Y can be expressed as finite sums of irreps

$$X = \bigoplus_{i \in I} m_i X^{(i)} \qquad Y = \bigoplus_{j \in J} n_j X^{(j)}$$

Let $L = I \cup J$. Then we can express X, Y by

$$X = \bigoplus_{\ell \in L} m_\ell X^{(\ell)} \qquad Y = \bigoplus_{\ell \in L} n_\ell X^{(\ell)}$$

where

$$m_\ell = \begin{cases} 1 & \text{if } \ell \in I, \\ 0 & \text{otherwise.} \end{cases} \qquad n_\ell = \begin{cases} 1 & \text{if } \ell \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Then by (2), for all $\ell \in L$,

$$\left\langle \chi, \chi^{(\ell)} \right\rangle = m_\ell = n_\ell = \left\langle \psi, \chi^{(i)} \right\rangle$$

hence $m_\ell = n_\ell$ for all $\ell \in L$. It follows that $X \cong Y$.

■

Example 2.6.1. Let $G = \mathcal{S}_3$. So far we know two irreps for G; the trivial rep, and the sign rep. Hence so far our character table looks like this:

	K_1	K_2	K_3
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	−1	1
$\chi^{(3)}$	?	?	?

Let χ^{def} be the character associated with the defining representation. By Maschke's Theorem, we know

$$\chi^{\text{def}} = m_1\chi^{(1)} + m_2\chi^{(2)} + m_3\chi^{(3)}$$

As we saw, $\chi^{\text{def}}(g) = \text{fix}(g)$ (the number of fixed points of g). Evaluating on the conjugacy classes, $\chi^{\text{def}}_{K_1} = 3$, $\chi^{\text{def}}_{K_2} = 1$, and $\chi^{\text{def}}_{K_3} = 0$. Hence

$$\begin{aligned}\left\langle \chi^{\text{def}}, \chi^{(1)} \right\rangle &= \frac{1}{|\mathcal{S}_3|}\sum_{i=1}^{3} \chi^{\text{def}}_{K_i}\chi^{(1)}_{K_i}|K_i| \\ &= \frac{1}{6}((3\cdot 1)\cdot 1 + (1\cdot 1)\cdot 3 + (0\cdot 1)\cdot 2) \\ &= 1\end{aligned}$$

So $m_1 = 1$. Similarly,

$$\begin{aligned}\left\langle \chi^{\text{def}}, \chi^{(2)} \right\rangle &= \frac{1}{|\mathcal{S}_3|}\sum_{i=1}^{3} \chi^{\text{def}}_{K_i}\chi^{(2)}_{K_i}|K_i| \\ &= \frac{1}{6}((3\cdot 1)\cdot 1 + (1\cdot -1)\cdot 3 + (0\cdot 1)\cdot 2) \\ &= 0\end{aligned}$$

hence $\chi^{\text{def}} = \chi^{(1)} + m_3\chi^{(3)}$. Now, observe that

$$\begin{aligned}\left\langle \chi^{\text{def}}, \chi^{\text{def}} \right\rangle &= \frac{1}{|\mathcal{S}_3|}\sum_{K} \left(\chi^{\text{def}}_{K}\right)^2 |K| \\ &= \frac{1}{6}(9\cdot 1 + 1\cdot 3 + 0\cdot 2) \\ &= \frac{12}{6} = 2\end{aligned}$$

by the corollary, we have

$$1 + m_3^2 = 2$$

so $m_3 = 1$. Hence, $\chi^{(3)} = \chi^{\text{def}} - \chi^{(1)} = \chi^{\text{def}} - 1$. Thus, the last row in our character table is

	K_1	K_2	K_3
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	-1	1
$\chi^{(3)}$	2	0	-1

2.7 Decomposition of the Group Algebra (Sagan 1.10)

We employ the machinery previously developed to decomposing the group algebra into irreducibles.

Proposition 2.7.1. *Let G be a finite group, and suppose $\mathbb{C}[\mathbf{G}] = \bigoplus_{i=1}^{k} m_i V^{(i)}$ where the $V^{(i)}$ are a complete list of pairwise inequivalent irreducible G-modules. Then*

(1) $m_i = \dim V^{(i)}$

(2) $\sum_i \left(\dim V^{(i)}\right)^2 = |G|$, *and*

(3) *The number of $V^{(i)}$ is the number of conjugacy classes of G.*

Proof. Let χ^{reg} be the character associated with $\mathbb{C}[\mathbf{G}]$, and let $\chi^{(i)}$ be the character associated to $V^{(i)}$.

(1) Applying the Corollary of the previous chapter, we have

$$\begin{aligned} m_i &= \left\langle \chi, \chi^{(i)} \right\rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi^{(i)}(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} |G|\delta_{g,\epsilon}\chi^{(i)}(g^{-1}) \\ &= \sum_{g \in G} \delta_{g,\epsilon}\chi^{(i)}(g^{-1}) \\ &= \chi^{(i)}(\epsilon) \end{aligned}$$

since $\chi^{(i)}(\epsilon) = \operatorname{tr} I_{d_i}$, we have $m_i = d_i$, as desired.

(2) Since we're using the regular representation, $\dim V = |G|$. Apply the previous result to obtain the claim.

(3) Recall that

$$k = \dim \mathbf{Z}_{\operatorname{End} \mathbb{C}[\mathbf{G}]}.$$

We first examine elements of $\mathbf{Z}_{\operatorname{End} \mathbb{C}[\mathbf{G}]}$. Recall that an endomorphism is an isomorphism whose domain and codomain are the same. That is, elements of $\mathbf{Z}_{\operatorname{End} \mathbb{C}[\mathbf{G}]}$ are of the form $\varphi : \mathbb{C}[\mathbf{G}] \to \mathbb{C}[\mathbf{G}]$.

Given, any $\mathbf{v} \in \mathbb{C}[\mathbf{G}]$, define the map $\varphi_{\mathbf{v}}$ by

$$\varphi_{\mathbf{v}}(\mathbf{w}) = \mathbf{w}\mathbf{v}.$$

Claim: $\mathbb{C}[\mathbf{G}] \cong \operatorname{End}_{\mathbb{C}[\mathbf{G}]}$ (as vector spaces)

Proof of Claim: Let $\phi : \mathbb{C}[\mathbf{G}] \hookrightarrow\!\!\!\!\rightarrow \operatorname{End}_{\mathbb{C}[\mathbf{G}]}$ be given by

$$\phi(\mathbf{v}) = \varphi_{\mathbf{v}}.$$

Let $\mathbf{v}, \mathbf{w} \in \mathbb{C}[\mathbf{G}]$ be arbitrary. Then $\phi(c(\mathbf{v} + \mathbf{w})) = c\varphi_{\mathbf{v}} + c\varphi_{\mathbf{w}}$. We now show ϕ is bijective.

(1) Let $\mathbf{v} \in \ker(\phi)$. Then $\varphi_{\mathbf{v}}$ is the 0 map. Hence, for all $\mathbf{w} \in \mathbb{C}[\mathbf{G}]$,

$$\begin{aligned} \varphi_{\mathbf{v}}(\mathbf{w}) &= \mathbf{w}\mathbf{v} \\ &= \mathbf{0} \end{aligned}$$

which occurs iff $\mathbf{v} = \mathbf{0}$.

(2) Let $\varphi \in \operatorname{End}_{\mathbb{C}[\mathbf{G}]}$. Then $\exists \mathbf{v} \in \mathbb{C}[\mathbf{G}]$ such that $\varphi(\boldsymbol{\epsilon}) = \mathbf{v}$. Then $\forall \mathbf{w} \in \mathbb{C}[\mathbf{G}]$,

$$\varphi(\mathbf{w}) = \varphi(\mathbf{w}\boldsymbol{\epsilon}) = \mathbf{w}\varphi(\boldsymbol{\epsilon}) = \mathbf{w}\mathbf{v}$$

where we pulled out the $\mathbf{w}$ by the fact that φ is a G-homomorphism. Hence, $\varphi = \varphi_{\mathbf{v}}$. It follows that φ is surjective.

As algebras, they are antiisomorphic. Hence, the centers are isomorphic. So

$$\mathbf{Z}_{\mathbb{C}[\mathbf{G}]} \cong \mathbf{Z}_{\operatorname{End} \mathbb{C}[\mathbf{G}]}$$

hence both are of dimension k. Finally, we examine elements of $\mathbf{Z}_{\mathbb{C}[\mathbf{G}]}$. Let $\mathbf{z} \in \mathbf{Z}_{\mathbb{C}[\mathbf{G}]}$ be arbitrarily chosen. Then $\forall \mathbf{h} \in \mathbf{G}$,

$$\begin{aligned} \mathbf{zh} &= \mathbf{hz} \\ \mathbf{h}^{-1}\mathbf{zh} &= \mathbf{z} \\ c_1\mathbf{h}^{-1}\mathbf{g}_1\mathbf{h} + c_2\mathbf{h}^{-1}\mathbf{g}_2\mathbf{h} + \cdots + c_n\mathbf{h}^{-1}\mathbf{g}_n\mathbf{h} &= c_1\mathbf{g}_1 + c_2\mathbf{g}_2 + \cdots + c_n\mathbf{g}_n \end{aligned}$$

Note that as $\mathbf{h}$ varies, $\mathbf{h}^{-1}\mathbf{g}_1\mathbf{h}$ runs over all of the conjugates of $\mathbf{g}_1$. Hence, if $\{\mathbf{g}_j\}_{j\in J}$ is the conjugacy class of $\mathbf{g}_1$, then $c_j = c_1$ for all $j \in J$. Hence, let

$$\mathbf{z}_i = \sum_{g \in K_i} g$$

for each conjugacy class K_i. Then $\mathbf{Z}_{\mathbb{C}[\mathbf{G}]}$ is spanned by the $\mathbf{z}_i$. Similarly, any linear combination of $\mathbf{z}_i$ is in $\mathbf{Z}_{\mathbb{C}[\mathbf{G}]}$. Linear independence follows because they are sums of disjoint collections of basis elements. The claim follows.

■

Proposition 2.7.2. *The irreducible characters of G form an orthonormal basis for the space of class functions $R(G)$.*

Proof. With respect to the G-invariant inner product, characters are orthonormal. Since $\dim R(G)$ is the number of conjugacy classes in V, the claim follows. ■

Theorem 2.7.3 (Character Relations of the Second Kind)**.** *Let K, L be conjugacy classes in G. Then*

$$\sum_{\chi} \chi_K \overline{\chi_L} = \frac{|G|}{|K|}\delta_{K,L}.$$

where the sum is over all irreducible characters of G.

Proof. By character relations of the first kind,

$$\frac{1}{|G|}\sum_{K} |K|\chi_K\overline{\chi_K} = \delta_{\chi,\psi}$$

but ■

3. Representations of $\mathcal{S}_n$

3.1 Young Subgroups, Tableaux, and Tabloids

3.2 Dominance and Lexicographic Ordering

3.3 Specht Modules (Sagan 2.3)

Note. Let $\lambda \vdash n$, and t be a tableau of shape $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$. Then if we let t' denote the "transpose" of t, i.e.

$$t_{i,j} = t'_{j,i} \qquad \text{valid } i, j$$

then we get another tableau of shape $\mu = (\mu_1, \mu_2, \dots, \mu_{\lambda_1})$, where

$$\mu_i = \# \text{ of } \lambda_j \text{ s.t. } \lambda_j \geq i.$$

It follows that $C_t \cong R_{t'}$. But $R_{t'}$ is a Young μ-subgroup, so C_t is isomorphic to a Young subgroup:

$$C_t \cong R_{t'} \cong \mathcal{S}_\mu.$$

Since $\mathcal{S}_\mu \leq \mathcal{S}_n$, this allows us to define the "sign" of elements of C_t in a "sensible" manner. Let $\pi \in \mathcal{S}_\mu$. Express π in terms of its factors:

$$\pi = (\pi_1, \pi_2, \dots, \pi_{\lambda_1}).$$

Then $\iota : \mathcal{S}_\mu \hookrightarrow \mathcal{S}_n$ defined by

$$\begin{aligned}\iota(\pi) &= \prod_{i=1}^{\lambda_1} \pi_i \\ &= \pi_1 \pi_2 \cdots \pi_{\lambda_1}\end{aligned}$$

is a well-defined homomorphism. Note that

$$\begin{aligned}\operatorname{sgn}(\iota(\pi)) &= \operatorname{sgn}\left(\prod_{i=1}^{\lambda_1}\right) \pi_i \\ &= \prod_{i=1}^{\lambda_i} \operatorname{sgn}(\pi_i).\end{aligned}$$

This motivates our next definition.

Definition 3.3.1. Let $\lambda \vdash n$ with $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$. Let t be a tableau of shape λ, and let $\pi \in C_t$:

$$\pi = (\pi_1, \pi_2, \dots, \pi_{\lambda_1}) \qquad \pi_i \in \mathcal{S}_{C_i}.$$

Then we define

$$\operatorname{sgn}(\pi) = \prod_{i=1}^{\lambda_i} \operatorname{sgn}(\pi_i).$$

We can now define κ_t.

Definition 3.3.2. Let λ, t be quantified as above. Then define

$$\kappa_t = C_t^- = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi.$$

Lemma 3.3.1. *Let $C_1, C_2, \ldots, C_{\lambda_1}$ denote the columns of t treated as their own Young Tableaux (i.e., 1 column, with the appropriate number of rows). Then*

$$\kappa_t = \prod_{i=1}^{\lambda_1} \kappa_{C_i}.$$

Proof. Recall that

$$\pi = (\pi_1, \pi_2, \ldots, \pi_{\lambda_1}).$$

Thus

$$\begin{aligned}
\kappa_t &= C_t^- \\
&= \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi \\
&= \sum_{\pi \in C_t} \left[\operatorname{sgn}\left(\prod_{i=1}^{\lambda_1} \pi_i\right) \prod_{i=1}^{\lambda_1} \pi_i\right] \\
&= \sum_{\pi \in C_t} \left[\left(\prod_{i=1}^{\lambda_1} \operatorname{sgn}(\pi_i)\right)\left(\prod_{i=1}^{\lambda_1} \pi_i\right)\right] \\
&= \sum_{\pi \in C_t} \left[\prod_{i=1}^{\lambda_1} \operatorname{sgn}(\pi_i)\pi_i\right] \\
&= \sum_{i=1}^{\lambda_1} \sum_{\pi_\alpha^{(i)} \in C_i} \left[\prod_{j=1}^{\lambda_1} \operatorname{sgn}\left(\pi_{\alpha,j}^{(i)}\right)\pi_{\alpha,j}^{(i)}\right] && (*) \\
&= \prod_{i=1}^{\lambda_1} \left[\sum_{\pi_\alpha^{(i)} \in C_i} \operatorname{sgn}\left(\pi_\alpha^{(i)}\right)\pi_\alpha^{(i)}\right] && (**) \\
&= \prod_{j=1}^{\lambda_1} \kappa_{C_j}
\end{aligned}$$

where we get from $(*)$ to $(**)$ as follows:

(a) Note that

$$\sum_{i=1}^{n_a} \sum_{j=1}^{n_b} a_i b_j = \prod_{i=a,b} \sum_{j=1}^{n_i} i_j$$

and in general,

$$\begin{aligned}
\sum_{i_1=1}^{n_1} \sum_{i_2=2}^{n_2} \cdots \sum_{i_m=1}^{n_m} \prod_{j=1}^{m} a_j &= \sum_{k=1}^{m} \sum_{i=1}^{n_k} \prod_{j=1}^{m} a_{i,j}^{(k)} \\
&= \prod_{j=1}^{m} \sum_{i=1}^{n_m} a_{i,j}^{(j)} \\
&=
\end{aligned}$$

□

Definition 3.3.3. If t is a tableau, then the associated *polytabloid* is

$$\boldsymbol{e}_t = \kappa_t \{\boldsymbol{t}\}.$$

Example 3.3.1. Suppose

$$t = \begin{array}{|ccc} \hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \hline \end{array}$$

Then

$$C_t = \mathcal{S}_{\{4,3\}} \times \mathcal{S}_{\{1,5\}} \times \mathcal{S}_{\{2\}}$$

thus

$$\begin{aligned} \kappa_t &= C_t^- \\ &= \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi \\ &= \end{aligned}$$

4. Appendix

4.1 List of Definitions

4.2 List of Theorems